ACTP **sheet nr. 02**

Exercise 1:

(a) Starting from $f^{abc} = -2i$ tr$\{[T^a, T^b]T^c\}$, show that f^{abc} is antisymmetric in $a \leftrightarrow b$, $a \leftrightarrow c$ and $b \leftrightarrow c$.

(b) Starting from the Jacobi identity, show that

$$f^{abd} f^{cde} + f^{bcd} f^{ade} + f^{cad} f^{bde} = 0 \,, \qquad a, b, c, e = 1, \ldots, \dim \,.$$

[Here the Einstein summation convention is used.]

(c) We define tr$\{T^a T^b T^c\} \equiv (d^{abc} + i f^{abc})/4$. Make use of the compleness relation from Exercise 4, to determine d^{aac} for SU(n).

Exercise 2: According to the lectures, every $A \in$ SU(2) can be expressed as $A = \exp(i\theta^a \sigma^a/2)$, where σ^a are the Pauli matrices. Verify that this parametrization leads to the same form of A as was found in the lecture through a parametrisation of SU(2) as S^3.

Exercise 3: Consider the Lie algebra of SO(3).

(a) Which properties do the generators T^a need to satisfy?

(b) Find an orthonormal basis for the algebra, normalized as tr$\{T^a T^b\} = \delta^{ab}/2$.

(c) What are the structure constants in this basis?

Exercise 4: Let T^a with $a = 1, \ldots, n^2 - 1$ be the generators of SU(n) [i.e. $(T^a)^\dagger = T^a$ and tr$\{T^a\} = 0 \quad \forall a$], normalized as tr$\{T^a T^b\} = \delta^{ab}/2$. Let furthermore $T^0 \equiv \mathbb{1}_{n\times n}/\sqrt{2n}$, and let M be a general complex $n \times n$ matrix.

(a) Show that $M = \sum_{A=0}^{n^2-1} C^A T^A$, where $C^A = 2$ tr$\{T^A M\} \in \mathbb{C}$.

(b) Making use of the result from point (a), verify that

$$\sum_{a=1}^{n^2-1} T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{n}\,\delta_{ij}\delta_{kl}\right) \,.$$