ACTP **sheet nr. 01**

Exercise 1: Show that

(a) the unit element e of a group is unique,

(b) for a given g, the inverse g^{-1} is unique.

Exercise 2: Show that

(a) $\dim \mathsf{SO}(n) = n(n-1)/2$,

(b) $\dim \mathsf{U}(n) = n^2$,

(c) $\dim \mathsf{SU}(n) = n^2 - 1$.

Exercise 3: Some manifolds (e.g. S^2) allow for no group structure, others for several independent ones. The space $\mathbb{R}^3$ with the normal addition is a group. Demonstrate that $\mathbb{R}^3$ is a group also with the "multiplication"

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) \equiv \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(y_1 x_2 - x_1 y_2) \right) .$$

Exercise 4: The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let $A \in \mathsf{SU}(2)$. Consider the mapping $A \to R(A)$, where R is a 3×3-matrix with

$$R_{ij} = \frac{1}{2} \operatorname{tr}\left[\sigma_i A \, \sigma_j A^\dagger\right] .$$

Show that $R \in \mathsf{SO}(3)$.

[*Hint:* Let $v, w \in \mathbb{R}^3$ and consider the transformation $v \mapsto v'$, $w \mapsto w'$, with $\sum_i v'_i \sigma_i \equiv A \left(\sum_j v_j \sigma_j\right) A^\dagger$ and $\sum_i w'_i \sigma_i \equiv A \left(\sum_j w_j \sigma_j\right) A^\dagger$. Show that $v', w' \in \mathbb{R}^3$ and $\sum_i v'_i w'_i = \sum_j v_j w_j$. Thereby the transformation is linear and orthogonal. Make use of $\operatorname{tr}[\sigma_i \sigma_j] = 2\delta_{ij}$, in order to project out the transformation matrix R. To show that $\det R = 1$, you may make use of the properties of the group manifold of $\mathsf{SU}(2)$.]