Standard Model	**Sheet 12**	13.05.2019

Exercise 1: Topological term. Because of the chiral anomaly, removing a complex phase from the quark mass matrix leads to a term in the QCD Lagrangian which is proportional to the "topological charge density", $c_s(x)$, defined as

$$c_s(x) := \frac{g_s^2}{64\pi^2}\,\epsilon^{\mu\nu\rho\sigma} H^A_{\mu\nu}(x) H^A_{\rho\sigma}(x)\;, \quad H^A_{\mu\nu} = \partial_\mu C^A_\nu - \partial_\nu C^A_\mu + g_s f^{ABD} C^B_\mu C^D_\nu\;.$$

Here $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor. Show that (for differentiable fields) $c_s(x)$ can be expressed as a total derivative:

$$\begin{aligned} c_s(x) &= \partial_\mu K^\mu(x)\;, \\ K^\mu(x) &= \frac{g_s^2}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\Big(C^A_\nu \partial_\rho C^A_\sigma + \frac{g_s}{3} f^{ABD} C^A_\nu C^B_\rho C^D_\sigma \Big)\;. \end{aligned}$$

[Therefore $c_s(x)$ does not contribute to the classical equations of motion; we might also naively expect that after the integration $\int \mathrm{d}^4x$ it does not contribute to the action.]

Exercise 2: Scale anomaly. On the classical level, setting quark masses to zero removes all scales from the QCD Lagrangian; we say that the theory is scale invariant, or "conformally invariant". This symmetry is explicitly broken by quantum corrections, which imply that the apparently dimensionless coupling constant actually depends on a scale.

(a) The "running coupling" of QCD, $g_s(Q_\mathrm{E})$, satisfies the equation

$$Q_\mathrm{E}\,\frac{\mathrm{d}}{\mathrm{d}Q_\mathrm{E}}\, g_s^2(Q_\mathrm{E}) = -2\, b_0\, g_s^4(Q_\mathrm{E})\;, \quad \text{for } Q_\mathrm{E} \gg 1\ \text{GeV}\;,$$

where $b_0 \equiv (11N_\mathrm{c} - 2N_\mathrm{f})/48\pi^2$ and $N_\mathrm{c} = N_\mathrm{f} = 3$. Find the general solution.

(b) One definition of a "QCD scale" comes from the asymptotic high-Q_E behaviour of g_s^2:

$$\Lambda_\mathrm{QCD} \equiv \lim_{Q_\mathrm{E}\to\infty} Q_\mathrm{E}\, \exp\left[-\frac{1}{2\,b_0\, g_s^2(Q_\mathrm{E})}\right]\;,$$

whereby $g_s^2(Q_\mathrm{E})$ is the solution from point (a). Assuming that experimental measurements show that $\alpha_s(M_Z) = g_s^2(M_Z)/4\pi \approx 0.12$, what do you get for Λ_QCD?

Exercise 3: Anomaly cancellation for general N_c.

(a) From Exercise 7.3, we know that for general N_c, the hypercharge assignments of quarks are

$$Y_{uL} = Y_{dL} = -\frac{1}{2N_\mathrm{c}}\;, \quad Y_{uR} = -\frac{N_\mathrm{c}+1}{2N_\mathrm{c}}\;, \quad Y_{dR} = \frac{N_\mathrm{c}-1}{2N_\mathrm{c}}\;.$$

Verify the cancellation of $\mathrm{SU_L(2)\times U_Y(1)}$ triangle gauge anomalies in this case.

(b) Like in Exercise 11.1, it can be shown that the electric charges of quarks (in units of e) are given by minus the right-handed hypercharge assignments: $Q_u = -Y_{uR}$, $Q_d = -Y_{dR}$. If a baryon is made out of N_u u-type quarks and $N_\mathrm{c} - N_u$ d-type quarks, show that its total electric charge is an integer only for odd N_c. [This happens to be the same constraint as is obtained from the cancellation of Witten's "global" $\mathrm{SU_L(2)}$ anomaly.]