Exercise 1: Massive gauge field propagator. In Exercise 1 of Sheet 2 a massless photon propagator was constructed. Let us now replace the gauge field A^μ by that of the Z-boson, Z^μ, and add a mass term to the Lagrangian,

$$\delta\mathscr{L} = \frac{1}{2} M_Z^2 \, Z^\mu Z_\mu \, .$$

Construct the corresponding propagator.

Exercise 2: A hidden invariance. In the script the Higgs doublet was assumed to be well approximated by a non-zero constant in the lower component, $\Phi = \frac{1}{\sqrt{2}}(0\ v)^T$, $v \in \mathbb{R}$, but the pattern of gauge field mass generation should remain the same also for a general Φ.

(a) Show that any Φ can be expressed as

$$\Phi = (\alpha_0 \sigma^0 + i\alpha_a \sigma^a) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} ,$$

where $v := \sqrt{2\,\Phi^\dagger \Phi} > 0$ and $A := \alpha_0 \sigma^0 + i\alpha_a \sigma^a$ is unitary ($\sigma^0 := \mathbb{1}$, $\alpha_0^2 + \sum_a \alpha_a^2 = 1$).

(b) Show that the matrices ${\sigma'}^a := A^\dagger \sigma^a A$ are traceless and Hermitean, and can therefore be expressed as linear combinations of the original σ^a.

(c) Show that if we define new $\mathsf{SU_L(2)}$ gauge fields through $\sum_a A_\mu^a {\sigma'}^a =: \sum_a A'^a_\mu \sigma^a$, then this transformation can be viewed as a rotation in the space of the A_μ^a.

(d) Armed with this knowledge, show finally that

$$\delta\mathscr{L} = \frac{1}{4} \Phi^\dagger (g_w A_\mu^a \sigma^a - g_Y B_\mu \sigma^0)(g_w A^{b\mu} \sigma^b - g_Y B^\mu \sigma^0) \Phi$$

necessarily leaves a certain linear combination of the gauge fields massless.

Exercise 3: Symmetry breaking with an adjoint scalar. According to Exercise 1(c) of Sheet 7, a gauge-invariant kinetic term for a scalar field in the adjoint representation reads

$$\mathscr{L} = \frac{1}{2} \operatorname{Tr} \left\{ [D_\mu, \Xi][D^\mu, \Xi] \right\} ,$$

where Ξ is a traceless and Hermitean matrix. Let D_μ be the same covariant derivative as acts on the Higgs doublet in the Minimal Standard Model, *viz.* $D_\mu = \partial_\mu - i g_w A_\mu^a T^a + i g_Y B_\mu T^0$. Suppose that Ξ develops an expectation value, e.g. $\Xi \to v\sigma^3$, $v > 0$. Show that in this case only two gauge field linear combinations obtain a mass, whereas two others remain massless.