Standard Model	**Sheet 7**	01.04.2019

Exercise 1: Non-Abelian and Abelian gauge transformations.

(a) A field (fermion or scalar) is said to transform under the "fundamental" representation of a non-Abelian gauge group SU(N) if $\Phi \to \Phi' = U\Phi$, where U is a gauge transformation matrix. The covariant derivative is written as $D_\mu = \partial_\mu - igA_\mu$, where g is a coupling constant and A_μ are traceless and Hermitean matrices. Show that for gauge "covariance", $D'_\mu \Phi' = UD_\mu\Phi$, the matrix A_μ needs to transform as

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{i}{g} U\partial_\mu U^{-1} \, .$$

(b) Show that if $U \in$ SU(N), then A'_μ is also traceless and Hermitean.

(c) A traceless and Hermitean matrix Ξ is said to transform under the "adjoint" representation of a gauge group SU(N) if $\Xi \to \Xi' = U\,\Xi\,U^{-1}$. Show that in this case it is the structure $[D_\mu, \Xi]$ which transforms covariantly: $[D'_\mu, \Xi'] = U[D_\mu, \Xi]U^{-1}$.

(d) Various MSM fields have different "charges" with respect to the Abelian $\mathrm{U_Y}(1)$ gauge group, whereas with respect to the non-Abelian groups all fields transform in the same way (or do not transform at all). Can you explain why different fields can have different couplings ($g_Y \to g_Y Y$) for the single gauge group $\mathrm{U_Y}(1)$?

Exercise 2: Conjugate Higgs field. A general SU(2) transformation can be written as

$$U = \sigma^0 \cos|\alpha| + i\sum_{a=1}^{3} \sigma^a \alpha^a \frac{\sin|\alpha|}{|\alpha|} \quad , \quad |\alpha| := \left(\textstyle\sum_{a=1}^{3} \alpha^a\alpha^a\right)^{1/2} \, .$$

Here σ^a, $a = 1, 2, 3$, are the Pauli matrices, α^a are real parameters, and $\sigma^0 := \mathbb{1}_{2\times 2}$.

(a) Verify that $U \in$ SU(2).

(b) If Φ transforms as $\Phi \to \Phi' = Ue^{ig_Y Y\theta}\Phi$, then how does $\tilde{\Phi} \equiv i\sigma^2\Phi^*$ transform?

Exercise 3: Hypercharge assignments. The fields Φ, Q_{iL}, u_{jR}, d_{jR}, L_{iL}, ν_{jR}, e_{jR} have the hypercharge assignments $-\frac{1}{2}$, $-\frac{1}{6}$, $-\frac{2}{3}$, $+\frac{1}{3}$, $+\frac{1}{2}$, 0, $+1$, respectively.

(a) Verify that the following Yukawa interactions are invariant both under $\mathrm{SU_L}(2)$ and $\mathrm{U_Y}(1)$:

$$\bar{Q}_{iL}\,\tilde{\Phi}\,u_{jR} \, , \quad \bar{Q}_{iL}\,\Phi\,d_{jR} \, , \quad \bar{L}_{iL}\,\tilde{\Phi}\,\nu_{jR} \, , \quad \bar{L}_{iL}\,\Phi\,e_{jR} \, .$$

(b) Leaving the assignments of Φ and the leptons fixed, but generalizing that of d_{jR} to be $(N_\mathrm{c} - 1)/(2N_\mathrm{c})$, where N_c is a free integer, determine the assignments of Q_{iL} and u_{jR}.