Standard Model	**Sheet 3**	04.03.2019

Exercise 1: On-shell spinors. Spin-$\frac{1}{2}$ particle and anti-particle spinors are defined through the momentum-space Dirac equations $(\not{P} - m)\, u(\vec{p}, s) = 0 = (\not{P} + m)\, v(\vec{p}, s)$, $s = \pm 1$.

(a) Show that these equations are satisfied by

$$u(\vec{p}, s) = C(\not{P} + m)\xi_s \ , \quad v(\vec{p}, s) = C'(\not{P} - m)\eta_s \ ,$$

where C, C' are normalization constants and ξ_s, η_s are so far arbitrary spinors.

(b) Show that the spinors are "orthogonal": $\bar{v}(\vec{p}, s)u(\vec{p}, s') = \bar{u}(\vec{p}, s)v(\vec{p}, s') = 0$.

(c) Choosing $|C|^2 = |C'|^2 = \frac{1}{E_p+m}$, $\sum_s (\xi_s)_\alpha(\bar{\xi}_s)_\beta = (\frac{\gamma^0+\mathbb{1}}{2})_{\alpha\beta}$, $\sum_s (\eta_s)_\alpha(\bar{\eta}_s)_\beta = (\frac{\gamma^0-\mathbb{1}}{2})_{\alpha\beta}$, verify the "completeness" relations:

$$\sum_{s=\pm 1} u_\alpha(\vec{p}, s)\bar{u}_\beta(\vec{p}, s) = (\not{P} + m)_{\alpha\beta} \ , \quad \sum_{s=\pm 1} v_\alpha(\vec{p}, s)\bar{v}_\beta(\vec{p}, s) = (\not{P} - m)_{\alpha\beta} \ .$$

[Hint: Show first that $(\not{P} + m)\gamma^0(\not{P} + m) = 2E_p(\not{P} + m)$.]

(d) Let us consider γ^0 in the "standard representation", $\gamma^0 :=$ diag$(\mathbb{1}_{2\times 2}, -\mathbb{1}_{2\times 2})$. Can you suggest representations for ξ_s, η_s so that the relations in (c) are satisfied?

Exercise 2: Crossing symmetry. Consider Mott scattering, i.e. the reaction $e^-(K_1, s_1) + \mu^-(K_2, s_2) \to e^-(P_1, t_1) + \mu^-(P_2, t_2)$ discussed in Exercise 3(a) of Sheet 2.

(a) By making use of Feynman rules, write down the invariant amplitude.

(b) Summing over the spins of initial and final states, derive the following expression:

$$\sum |\mathscr{M}|^2 = \frac{e^4}{(K_1 - P_1)^4} \mathrm{Tr}\, [\gamma^\alpha(\not{K}_1 + m_e)\gamma^\beta(\not{P}_1 + m_e)] \mathrm{Tr}\, [\gamma_\alpha(\not{K}_2 + m_\mu)\gamma_\beta(\not{P}_2 + m_\mu)] \ .$$

(c) Comparing with the corresponding expression in the script for $e^-e^+ \to \mu^-\mu^+$, explain how the kinematic invariants s, t, u need to be "crossed" (exchanged with each other) in order to read off the final result without redoing any contractions.

(d) In "Rutherford scattering" an electron scatters off a proton, $e^-p^+ \to e^-p^+$. How can one obtain the cross section for this reaction from that for $e^-\mu^- \to e^-\mu^-$?

Exercise 3: Total cross section. In the script the following cross section is derived for $e^-e^+ \to \mu^-\mu^+$ scattering in the center-of-mass frame:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha_{em}^2}{2s^3}\sqrt{\frac{s - 4m_\mu^2}{s - 4m_e^2}}\, \Big[(m_e^2 + m_\mu^2 - u)^2 + (m_e^2 + m_\mu^2 - t)^2 + 2(m_e^2 + m_\mu^2)s\Big]\theta(\sqrt{s} - 2m_\mu) \ .$$

Derive the corresponding total cross section. $[\frac{4\pi\alpha_{em}^2}{3s}(\frac{s-4m_\mu^2}{s-4m_e^2})^{\frac{1}{2}}(1+\frac{2m_e^2}{s})(1+\frac{2m_\mu^2}{s})\theta(\sqrt{s}-2m_\mu).]$

Exercise 4: Massless limit. Show that $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ of Exercise 3 reduces to $\frac{\alpha_{em}^2}{4s}(1 + \cos^2\theta)$ for $\sqrt{s} \gg 2m_\mu$, where θ is the scattering angle, and sketch the physical meaning of this result. Verify also the classic formula for the total cross section: $\sigma = \frac{4\pi\alpha_{em}^2}{3s}$.