5.2 Neutrino masses

Neutrino masses are much smaller than all other fermion masses, and until the 1990's were consistent with zero. Therefore the fact that at least two neutrinos are now known to possess masses is often referred to as „Beyond the Standard Model" physics, even though a proper treatment only requires a minor addition to the Lagrangian of p. 49.

The quadratic neutrino part from p.49, with $\tilde{\Phi} \to \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}$ (cf. p.35):

$$\delta\mathcal{L} = \sum_{k=1}^{3}\left\{ \bar{\nu}_{kL}\, i\gamma^\mu \partial_\mu \nu_{kL} + \bar{\nu}_{kR}\, i\gamma^\mu\partial_\mu \nu_{kR}\right\} - \frac{v}{\sqrt{2}} \sum_{k,l=1}^{3} \underbrace{\left\{ h_\nu^{kl}\, \bar{\nu}_{kL}\, \nu_{lR} + \text{H.c.}\right\}}_{\text{„Dirac-type mass term"}}$$

Moreover ν_R's do not transform <u>at all</u> in gauge transformations.

<u>Two-component spinors:</u> In order to better understand the structure, it is advantageous to use the <u>Weyl representation</u> of Dirac matrices:

$$\gamma^0 := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} ; \quad \gamma^k := \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} ; \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

p.13

Subsequently

$$P_L = \frac{\mathbb{1}-\gamma_5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \frac{\mathbb{1}+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

$$\nu = (P_L + P_R)\,\nu = \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix}.$$

These are now 2-component spinors!

Then $\gamma^0\gamma^\mu = \left\{ \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \right\} =: \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{pmatrix}$, and

$$\delta\mathcal{L} = \nu_{kL}^\dagger\, i\sigma^\mu\partial_\mu \nu_{kL} + \nu_{kR}^\dagger\, i\bar{\sigma}^\mu \partial_\mu \nu_{kR} - \left\{ M_D^{kl}\, \nu_{kL}^\dagger\, \nu_{lR} + \text{H.c.} \right\} ; \quad M_D^{kl} := \frac{h_\nu^{kl} v}{\sqrt{2}}.$$

<u>Claim:</u> A further addition,

$$\delta\mathcal{L} = \frac{1}{2}\left\{ M_M^{kl}\, \nu_{kR}^T\, i\sigma_2\, \nu_{lR} + \text{H.c.}\right\},$$

is both gauge-invariant and invariant in $L_+^\uparrow$ (p.13). The couplings M_M^{kl} form a „<u>Majorana-type mass term</u>".

<u>Proof:</u> Gauge invariance is trivial because ν_{lR} is singlet. For Lorentz-invariance we may recall that $L_+^\uparrow$ is homomorph to $SL(2,\mathbb{C})$ (cf. „Classical field theory"). Focussing on spin indices (i.e. drop k,l), we see that

$$i\sigma_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: \varepsilon^{\alpha\beta}$$

$$\Rightarrow \quad \nu_R^T\, i\sigma_2\, \nu_R = \nu_{R\alpha}\, \varepsilon^{\alpha\beta}\, \nu_{R\beta}$$

$\nu_{R\alpha} \to M_\alpha^{\ \alpha'}\, \nu_{R\alpha'}$

$$\Rightarrow \quad \varepsilon^{\alpha\beta} M_\alpha^{\ \alpha'} M_\beta^{\ \beta'}\, \nu_{R\alpha'}\, \nu_{R\beta'} \overset{!}{=} \underbrace{\det(M)}_{1}\, \varepsilon^{\alpha'\beta'}\, \nu_{R\alpha'}\, \nu_{R\beta'} \quad \square .$$

Back to four-component spinors:

What is the meaning of $i\sigma_2$?

Recall from Exercise 4.2:

$$C = \hat{\gamma}_0 \hat{\gamma}_2 = -i\gamma^0\gamma^2 = -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.$$

charge conjugation matrix

In the literature C is often defined with an opposite sign.

Recall from p. 15:

$$\psi^c := C\bar{\psi}^T = C(\psi^\dagger\gamma^0)^T = C\gamma^{0T}\psi^* ,$$

$$\bar{\psi}^c = (C\gamma^{0T}\psi^*)^\dagger\gamma^0 = \psi^T\gamma^{0*}C^\dagger\gamma^0 = -\psi^T\gamma^0 C\gamma^0$$

$$= \psi^T C\gamma^0\gamma^0 = \psi^T C .$$

Note also:

$$(\psi^c)_R = \frac{\mathbb{1}+\gamma_5}{2}\psi^c = \frac{1+\gamma_5}{2}C\gamma^{0T}\psi^* = C\frac{1+\gamma_5}{2}\gamma^{0T}\psi^*$$

$$= C\gamma^{0T}\frac{1-\gamma_5}{2}\psi^* = (\psi_L)^c ,$$

i.e. right-handed fields can be represented as charge conjugates of left-handed fields, and vica versa ⇒ p. 52.

So now:

$$\underbrace{\nu^T_{kR}\, i\sigma_2\, \nu_{\ell R}}_{\text{2-component}} = \underbrace{-\nu^T_{kR}\, C\, \nu_{\ell R}}_{\text{4-component}} = -\bar{\nu}^c_{kR}\,\nu_{\ell R} ,$$

and consequently

$$\delta\mathcal{L} = -\frac{1}{2}\left\{ M_M^{k\ell}\,\bar{\nu}^c_{kR}\nu_{\ell R} + \text{H.c.}\right\} .$$

Another convention:

Sometimes so-called „Majorana spinors" are used:

$$\tilde{\nu}_R := \begin{pmatrix} i\sigma_2\nu_R^* \\ \nu_R \end{pmatrix}$$

$$\Rightarrow \tilde{\nu}^c_R = C\gamma^{0T}\tilde{\nu}^*_R = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} i\sigma_2\nu_R \\ \nu_R^* \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}\begin{pmatrix} \nu_R^* \\ i\sigma_2\nu_R \end{pmatrix} = \begin{pmatrix} i\sigma_2\nu_R^* \\ \nu_R \end{pmatrix} \stackrel{!}{=} \tilde{\nu}_R .$$

We say that a Majorana fermion „is its own antiparticle".

We may note that

$$(i\sigma_2\nu_R^*)^\dagger\, i\bar{\sigma}^\mu\partial_\mu\,(i\sigma_2\nu_R^*) = \nu_R^T\, i\overbrace{\sigma_2\bar{\sigma}^\mu\sigma_2}^{\sigma^{\mu T}}\,\partial_\mu\,\nu_R^*$$

$$\doteq -\nu_R^\dagger\, i\bar{\sigma}^\mu\overleftarrow{\partial}_\mu\,\nu_R = \nu_R^\dagger\, i\bar{\sigma}^\mu\partial_\mu\,\nu_R ,$$

partial integration

$$c_i A^T_{ij} c_j \underset{\text{Grassmann}}{=} -c_j A^T_{ij} c_i \underset{\text{transpose}}{=} -c_j A_{ji} c_i$$

(see p. 15 for more details)

and that

$$\bar{\tilde{\nu}}_{kR}\tilde{\nu}_{\ell R} = (-\nu^T_{kR}i\sigma_2 \;\; \nu^\dagger_{kR})\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} i\sigma_2\nu^*_{\ell R} \\ \nu_{\ell R} \end{pmatrix}$$

$$= -\nu^T_{kR}i\sigma_2\nu_{\ell R} + \nu^\dagger_{kR}i\sigma_2\nu^*_{\ell R}$$

$$= -\nu^T_{kR}i\sigma_2\nu_{\ell R} - [\nu^T_{\ell R}i\sigma_2\nu_{kR}]^\dagger .$$

So, if $(M_M^{\ell k})^* = M_M^{k\ell}$ (cf. p. 55), then $\delta\mathcal{L}$ from p. 53 can be expressed as

$$\delta\mathcal{L} = \bar{\nu}_{kL}\, i\gamma^\mu\partial_\mu\,\nu_{kL} + \frac{1}{2}\bar{\tilde{\nu}}_{kR}\, i\gamma^\mu\partial_\mu\,\tilde{\nu}_{kR}$$

$$-\{M_D^{k\ell}\,\bar{\nu}_{kL}\tilde{\nu}_{\ell R} + \text{H.c.}\} - \frac{1}{2}M_M^{k\ell}\,\bar{\tilde{\nu}}_{kR}\tilde{\nu}_{\ell R} .$$

See-saw mechanism:

(P. Minkowski, Phys. Lett. B 67 (1977) 421; M. Gell-Mann, P. Ramond, R. Slasky 1979; T. Yanagida, Prog. Theor. Phys. 64 (1980) 1103.)

Although based on convoluted mathematical considerations, the existence of Majorana masses for right-handed neutrinos leads to many important physics consequences.

One of these is that if we <u>assume</u> the right-handed neutrinos to be very heavy, $M_M^{kl} \sim M_X$ (p. 51), then it appears „natural" that the observed left-handed neutrinos are light. This is due to the <u>„see-saw mechanism"</u>.

Step (i): The matrix M_M^{kl} can be assumed real, diagonal, and positive.

The matrix must be symmetric:

$$\nu_{kR}^T i\sigma_2 \nu_{lR} = \underbrace{(\nu_{kR})_\alpha}_{\text{Grassmann}} \underbrace{(i\sigma_2)_{\alpha\beta}}_{\text{antisymmetric}} (\nu_{lR})_\beta$$

$$= (\nu_{lR})_{\cancel{\beta}\,\alpha} (i\sigma_2)_{\cancel{\beta\alpha}\,\alpha\beta} (\nu_{kR})_{\cancel{\alpha}\,\beta} = \nu_{lR}^T i\sigma_2 \nu_{kR}.$$

It can thus be diagonalized. Any remaining phases can be removed:

$$M_M^{kk} = |M_M^{kk}| e^{i\phi_{kk}} \Rightarrow \nu_{kR} \to e^{-\frac{i\phi_{kk}}{2}} \nu_{kR} \Rightarrow \Box.$$

Step (ii): Determine the „propagator" of a very heavy ν_{kR}:

$$\delta\mathcal{L} = \frac{1}{2} M_M^{kk} \Big\{ \underbrace{(\nu_{kR})_\alpha \varepsilon^{\alpha\beta} (\nu_{kR})_\beta}_{\nu_{kR}^T i\sigma_2 \nu_{kR}} - \underbrace{(\nu_{kR}^*)_\alpha \varepsilon^{\alpha\beta} (\nu_{kR}^*)_\beta}_{+\,\nu_{kR}^\dagger(-i\sigma_2)\nu_{kR}^*} \Big\}$$

$$= \frac{1}{2} M_M^{kk} \begin{pmatrix} \nu_{kR1} \\ \nu_{kR2} \\ \nu_{kR1}^* \\ \nu_{kR2}^* \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_{kR1} \\ \nu_{kR2} \\ \nu_{kR1}^* \\ \nu_{kR2}^* \end{pmatrix}$$

$$\Rightarrow \text{inverse matrix} = \frac{2}{M_M^{kk}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\overset{*}{\Rightarrow} -\langle (\nu_{kR})_\alpha (\nu_{kR})_\beta \rangle = \langle (\nu_{kR}^*)_\alpha (\nu_{kR}^*)_\beta \rangle = \frac{i}{M_M^{kk}} \varepsilon^{\alpha\beta}.$$

*To understand the precise prefactors one needs to carry out Gaussian Grassmann integrals.

Step (iii): „Integrate out" the very heavy ν_{kR}:

$$\Big\langle e^{-i\int_x (M_D^{kl} \nu_{kL}^\dagger \nu_{lR} + M_D^{*kl} \nu_{lR}^\dagger \nu_{kL})} \Big\rangle_{\nu_R}$$

$$= \Big\langle 1 - \frac{1}{2}\int_{x,y} \Big(M_D^{kl} M_D^{pq} \nu_{kL\alpha}^* \underbrace{\nu_{lR\alpha} \nu_{pL\beta}^*}\, \nu_{qR\beta} + M_D^{*kl} M_D^{*pq} \underbrace{\nu_{lR\alpha}^* \nu_{kL\alpha}}\, \nu_{qR\beta}^* \nu_{pL\beta} \Big) + \ldots \Big\rangle_{\nu_R}$$

$$= 1 + \frac{1}{2}\frac{i}{M_M^{ll}} \int_x \Big(M_D^{kl} M_D^{pl} \nu_{kL\alpha}^* \varepsilon^{\alpha\beta} \nu_{pL\beta}^* - M_D^{*kl}_{\;p} M_D^{*pl}_{\;k} \nu_{\cancel{k}L\alpha\,p}\, \varepsilon^{\alpha\beta} \nu_{\cancel{p}L\beta\,k} \Big) + \ldots$$

$$\approx \exp\Big\{ i\int_x \Big[\frac{1}{2} \sum_l \frac{M_D^{kl} M_D^{pl}}{M_M^{ll}} \nu_{kL}^\dagger i\sigma_2 \nu_{pL}^* + \text{H.c.} \Big] \Big\}.$$

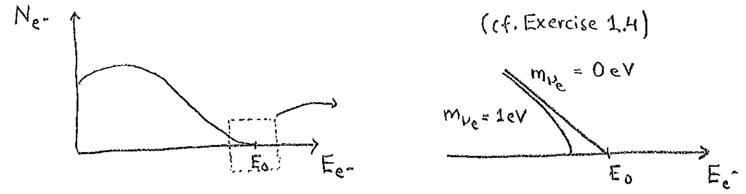
$$\uparrow$$

This is the „see-saw": if $M_D^{kl} = \frac{h_\nu^{kl} v}{\sqrt{2}} \sim 1$ GeV, $M_M^{ll} \sim 10^{15}$ GeV, then the effective left-handed neutrino masses are $\sim \frac{1^2}{10^{15}}$ GeV $\sim 10^{-6}$ eV !

This is a „see-saw": one is „heavy", the other is „light".

Basics of neutrino oscillations:

It is very difficult to measure the absolute value of a neutrino mass; consider tritium β-decay, ${}^3H \to {}^3He + e^- + \bar{\nu}_e$:

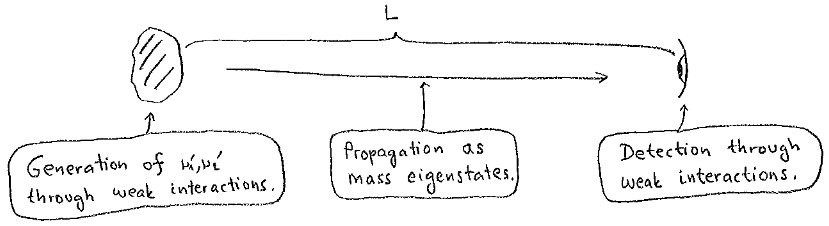


In contrast, mass differences are experimentally accessible. As an example, consider Cabibbo-type mixing with two generations. Like on p. 19:

$$\begin{pmatrix} \nu_1' \\ \nu_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

weak interaction eigenstates

mass eigenstates



Suppose that the situation is stationary, i.e. energy is fixed.

$$\Rightarrow \quad |\nu_1'(t,L)\rangle = \cos\theta\,|\nu_1\rangle\, e^{-iEt + ip_1L} + \sin\theta\,|\nu_2\rangle\, e^{-iEt + ip_2L};$$

$$E = \sqrt{p_1^2 + m_1^2} = \sqrt{p_2^2 + m_2^2};$$

$$\langle \nu_2'| = -\sin\theta \langle \nu_1| + \cos\theta \langle \nu_2|$$

$$\Rightarrow \quad \langle \nu_2' | \nu_1'(t,L)\rangle = \cos\theta \sin\theta\, e^{-iEt}\left(e^{ip_2L} - e^{ip_1L}\right)$$

$$= \sin 2\theta \cdot e^{-iEt} \cdot e^{i\left(\frac{p_1+p_2}{2}\right)L} \cdot \frac{1}{2}\left(e^{i\frac{p_2-p_1}{2}L} - e^{-i\frac{p_2-p_1}{2}L}\right)$$

$$= i \sin 2\theta \cdot e^{-iEt} \cdot e^{i\left(\frac{p_1+p_2}{2}\right)L} \cdot \sin\left(\frac{p_2-p_1}{2}L\right)$$

$$\Rightarrow \quad P(1' \to 2') = |\langle \nu_2' | \nu_1'(t,L)\rangle|^2$$

$$= \sin^2(2\theta)\, \sin^2\left(\frac{p_2-p_1}{2}L\right).$$

For $E \gg m_i$: $\quad p_i = \sqrt{E^2 - m_i^2} = E - \frac{1}{2}\frac{m_i^2}{E} + \dots$

$$\Rightarrow \quad P(1' \to 2') = \sin^2(2\theta)\, \sin^2\left(\frac{L \cdot \Delta m^2}{4E}\right).$$

Inserting units: $\quad \dfrac{L\,\Delta m^2}{4E} = \dfrac{L\,[km] \cdot \Delta m^2\,[eV^2]}{E\,[GeV]} \cdot \dfrac{10^3 m \cdot eV^2}{4\,GeV}$

$$\frac{10^{18} fm \cdot GeV}{4 \cdot 10^{18}} \underset{p.2}{\approx} \frac{5}{4}$$

So even if $|\Delta m^2| \ll eV^2$, something can be seen, provided that $L \gg km$.

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