5. Beyond the Standard Model

5.1 Grand Unification

We start by collecting together all parts of the Standard Model, and suggest then motivations for searching for a more complete theory. Subsequently experimental tests and theoretical consequences of such models are outlined.

Standard Model in weak interaction eigenbasis:
(we employ once again unprimed fields, cf. p. 28)

* Spin-1: Gauge bosons of $SU_c(3) \times SU_L(2) \times U_Y(1)$.
  - $U_Y(1)$: B_μ
  - $SU_L(2)$: A^a_μ, $a = 1,\dots,3$; $3 = 2^2-1$.
  - $SU_c(3)$: C^A_μ, $A = 1,\dots,8$; $8 = 3^2-1$.

$$\delta\mathcal{L} = \underset{(p.41)}{-\frac{1}{4} H^A_{\mu\nu} H^{A\mu\nu}} - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \underset{(p.5)}{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}} + \underset{(p.47)}{\mathcal{L}_{\text{anomaly}}} .$$

* Spin-$\frac{1}{2}$: Fermions transforming under a specific (anomaly-free) representation of $SU_c(3) \times SU_L(2) \times U_Y(1)$:

"irrep" := irreducible representation

fermion type	dimension of $SU_c(3)$-irrep	dimension of $SU_L(2)$-irrep	hypercharge assignment
$\begin{pmatrix} \nu \\ e \end{pmatrix}_L =: L_L$	1	2	$+\frac{1}{2}$
ν_R	1	1	0
e_R	1	1	$+1$
$\begin{pmatrix} u \\ d \end{pmatrix}_L =: Q_L$	3	2	$-\frac{1}{6}$
u_R	3	1	$-\frac{2}{3}$
d_R	3	1	$+\frac{1}{3}$

Of each type there are three copies, "generations".

$$\delta\mathcal{L} = \sum_{k=1}^{3} \sum_{\Psi} \bar{\Psi}_k i\gamma^\mu D_\mu \Psi_k \; ; \quad D_\mu = \mathbb{1}\partial_\mu - i g_s T^A_R C^A_\mu - i g_w T^a_R A^a_\mu - i g_Y Y_R B_\mu .$$

generations (k); fermion types (Ψ)

unit matrix of dimension $\dim(T^A_R) \times \dim(T^a_R)$ ($\mathbb{1}$)

generators for the chosen irrep (T^A_R, T^a_R, Y_R)

* Spin-0: Higgs field Φ, with assignments $(1, 2, -\frac{1}{2})$.

$$\delta\mathcal{L} = \underset{(p.27)}{(D_\mu \Phi)^\dagger (D^\mu \Phi)} \underset{(p.33)}{- V(\Phi^\dagger\Phi)} + \underset{(p.35)}{\mathcal{L}_{\text{Yukawa}}} \; ;$$

$$\mathcal{L}_{\text{Yukawa}} = - \sum_{k,l=1}^{3} \Big[\bar{Q}_{kL} \left(h_u^{kl} \tilde{\Phi} u_{lR} + h_d^{kl} \Phi d_{lR} \right) + \bar{L}_{kL} \left(h_\nu^{kl} \tilde{\Phi} \nu_{lR} + h_e^{kl} \Phi e_{lR} \right) + \text{H.c.} \Big] .$$

generations (k, l)

Items on a wishlist for a simpler theory:

(i) Could the gauge group $SU_c(3) \times SU_L(2) \times U_Y(1)$ be „unified" into a single group? In particular, could the couplings g_s, g_w, g_Y be all proportional to a single g?

(ii) Could the fermions be assembled into fewer „multiplets"?

(iii) Is there a way to bring structure into the Yukawa interactions? In particular, can the masses of different particles be related to each other?

Possible answers:

(i) * Number of gauge bosons in the Standard Model: $8+3+1 = 12$. The new group should have at least as many.

* If the group is $SU(N)$, its generators are traceless. So, there are $N-1$ independent diagonal generators. This should be at least as many as there are independent diagonal generators in $SU_c(3) \times SU_L(2) \times U_Y(1)$: $2+1+1 = 4$. So, $SU(5)$ is the simplest possibility.

The SU(5) model was proposed by H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438.

* With $SU(5)$, there are $5^2 - 1 = 24$ gauge bosons. So, we need a Higgs mechanism which makes 12 of them very heavy. But the Standard Model ones (particularly gluons and photons) should remain light. So, the Higgs needs to lie in a suitable representation.

(ii) Let us count the number of chiral fermions (per generation):

$$\underbrace{\nu_L, e_L, \nu_R, e_R}_{4} + \underbrace{3 \times (u_L, d_L, u_R, d_R)}_{12} = 16.$$

Traditionally ν_R was not considered part of the Standard Model $\Rightarrow$ 15.
Clearly $5 \neq 15, 16$, but:

Lemma: If an N-component vector v_i transforms as $v_i \to U_i{}^j v_j$, then symmetric and antisymmetric 2-component tensors s_{ij}, a_{ij} form representations of dimensions $\frac{N(N+1)}{2}$ and $\frac{N(N-1)}{2}$, respectively.

Proof: The symmetry property is conserved:

$$s'_{ji} = U_j{}^k U_i{}^\ell s_{k\ell} = U_j{}^k U_i{}^\ell s_{\ell k}$$
$$= U_i{}^{\overset{k}{\ell}} U_j{}^{\overset{\ell}{k}} s_{\underset{k\ell}{\ell k}} = s'_{ij},$$

and similarly with a_{ij}. Dimensions can be obtained like for an $N \times N$-matrix.

For 16 chiral fermions a suitable irrep can be found with SO(10); H. Georgi 1974; H. Fritzsch & P. Minkowski 1975.

Put $N=5 \Rightarrow \frac{N(N+1)}{2} = 15$, $\frac{N(N-1)}{2} = 10$.
(In the end $15 \to \bar{5} \oplus 10$ turns out to be preferable.)

(iii) We do not get rid of generations with $SU(5)$, but there are less Yukawa matrices — $h_{\bar{5}}^{k\ell}, h_{10}^{k\ell}$ instead of $h_u^{k\ell}, h_d^{k\ell}, h_\nu^{k\ell}, h_e^{k\ell}$.

More details on the Higgs mechanism

Since only a partial vector boson mass generation is desirable, it is best to put the Higgs scalar in the adjoint representation (cf. Exercise 8.3):

$$\Xi = (\text{traceless \& Hermitean } 5\times 5 \text{ matrix}) =: \underbrace{\Xi^a T^a}_{5^2-1=24 \text{ generators}}$$

For SU(N), Ξ has N-1 independent entries after diagonalization. Therefore, $\mathrm{Tr}[\Xi^2]$, $\mathrm{Tr}[\Xi^3]$, …, $\mathrm{Tr}[\Xi^N]$ are independent structures. Odd powers can be excluded by postulating a symmetry.

Gauge transformation: $\Xi \to \Xi' = U\,\Xi\,U^{-1}$

$$\Rightarrow \quad V(\Xi) = -\mu^2\,\mathrm{Tr}[\Xi^2] + \lambda_1\left(\mathrm{Tr}[\Xi^2]\right)^2 + \lambda_2\,\mathrm{Tr}[\Xi^4]\,.$$

If we parametrize Ξ as $\Xi = \begin{pmatrix} v_1 & & & & 0 \\ & v_2 & & & \\ & & v_3 & & \\ & & & v_4 & \\ 0 & & & & -v_1-v_2-v_3-v_4 \end{pmatrix}$,

then in general several minima can be found (cf. Exercise 13.3).

Let us assume that a minimum with the structure

$$\Xi_0 := \begin{pmatrix} v & & & & \\ & v & & & \\ & & v & & \\ & & & -\frac{3v}{2} & \\ & & & & -\frac{3v}{2} \end{pmatrix}$$

is a global one. What kind of masses are given to gauge bosons?

Exercise 8.3 $\Rightarrow$ $\delta\mathcal{L} = \frac{1}{2}\mathrm{Tr}\left\{[D_\mu, \Xi][D^\mu, \Xi]\right\}$; $D_\mu =: \partial_\mu - igT^aE^a_\mu$.

* 8 generators with the structure $T^a = \begin{pmatrix} T^a_{3\times 3} & 0 \\ 0 & 0 \end{pmatrix}$

  $\Rightarrow [T^a, \Xi_0] = 0 \Rightarrow$ gluons remain massless.

* 3 generators with the structure $T^a = \begin{pmatrix} 0 & 0 \\ 0 & T^a_{2\times 2} \end{pmatrix}$

  $\Rightarrow [T^a, \Xi_0] = 0 \Rightarrow W^\pm, Z^0$ remain massless.

To construct a realistic theory, another Higgs field is introduced too, which then „breaks" $SU_L(2)$.

* 1 generator with the structure $T^a \propto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & -3/2 \end{pmatrix}$

  $\Rightarrow [T^a, \Xi_0] = 0 \Rightarrow$ photon remains massless,

* The remaining 12 generators span the 12 „off-diagonal" entries:

$$T^aE^a_\mu =: \begin{pmatrix} & \emptyset_{3\times 3} & & X^1_\mu & Y^1_\mu \\ & & & X^2_\mu & Y^2_\mu \\ & & & X^3_\mu & Y^3_\mu \\ X^{1*}_\mu & X^{2*}_\mu & X^{3*}_\mu & \multicolumn{2}{c}{\emptyset_{2\times 2}} \\ Y^{1*}_\mu & Y^{2*}_\mu & Y^{3*}_\mu & & \end{pmatrix}.$$

The „X,Y"-bosons mediate „charged currents" like $W^\pm$, and are massive:

$$\left[\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \Xi_0\right] = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & -\frac{3v}{2} \end{pmatrix} - \begin{pmatrix} v & 0 \\ 0 & -\frac{3v}{2} \end{pmatrix}\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{3v}{2}X \\ vX^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & vX \\ -\frac{3v}{2}X^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{5v}{2}X \\ \frac{5v}{2}X^* & 0 \end{pmatrix}$$

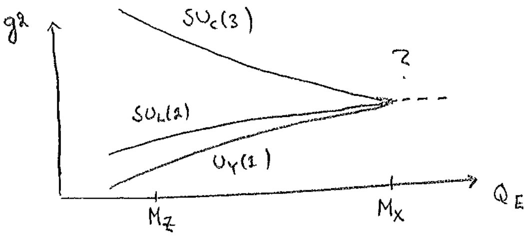
$$\Rightarrow \frac{1}{2}\mathrm{Tr}\left\{[-igT^aE^a_\mu, \Xi_0][-igT^bE^{b\mu}, \Xi_0]\right\} = -\frac{g^2}{2}\mathrm{Tr}\begin{pmatrix} 0 & -\frac{5v}{2}X \\ \frac{5v}{2}X^* & 0 \end{pmatrix}\begin{pmatrix} 0 & -\frac{5v}{2}X \\ \frac{5v}{2}X^* & 0 \end{pmatrix}$$

$$= \underbrace{\frac{25}{4}g^2v^2}_{M_X^2 = M_Y^2}\sum_{\alpha=1}^{3}\left(X^{*\alpha}_\mu X^{\alpha\mu} + Y^{*\alpha}_\mu Y^{\alpha\mu}\right).$$

Some consequences of Grand Unification

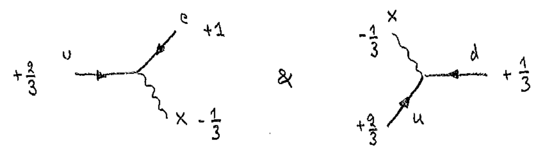
As always in physics, theoretical constructions like Grand Unification remain on the level of speculation, unless they lead to „signature" consequences which can in the end be experimentally verified. Some possibilities:

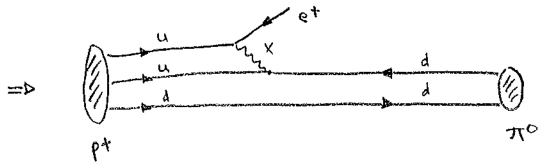
(i) There is only one gauge coupling, g. So, if we „run" (Exercise 12.2) the known couplings to high virtualities, at which the „symmetry breaking scale" $\sim M_X$ plays no role, the couplings should „unify":



(ii) There are necessarily quarks and leptons in the same multiplet, so the „charged currents" mediated by X, Y can transform them to each other:

Since the construction of GUTs relies on chiral fields (p.50), and we want to assemble them in a single multiplet, charge conjugation (p.15) is used for transforming R to L. Consequently particle and antiparticle „arrow directions" are no longer conserved.





$p^+ \to e^+\pi^0$.

The order of magnitude of the decay rate can easily be estimated (Exercise 13.1), and leads to a lower bound on M_X.

(iii) Renormalizability restricted the types of operators that can appear in the Standard Model (p.26). But now it is enough that the „fundamental theory", GUT, be renormalizable; Standard Model is to be viewed as a „low-energy effective theory". This means that new operators are admissible, as long as they are suppressed by M_X and are gauge-invariant. This leads to many new processes.

S. Weinberg, Phys. Rev. Lett. 43 (1979) 1566; F. Wilczek and A. Zee, Phys. Rev. Lett. 43 (1979) 1571.

(iv) If $U(1)_{em}$ is a consequence of the „breaking" of a „simple" group like SU(5), then the theory normally predicts the existence of magnetic monopoles. Their non-observation again sets a lower bound on M_X (the monopole mass is $\propto M_X$).

G. 't Hooft, Nucl. Phys. B 79 (1974) 276; A.M. Polyakov, JETP Lett. 20 (1974) 194.

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