3.7 Quark mixing, flavour physics

Concentrating on quarks for now, the starting point is (p.28, 36)

$$\delta\mathcal{L} = (\bar{Q}'_{1L}\,\bar{Q}'_{2L}\,\bar{Q}'_{3L})\, i\gamma^\mu D^{(L)}_\mu \begin{pmatrix} Q'_{1L} \\ Q'_{2L} \\ Q'_{3L} \end{pmatrix} + (\bar{u}'_R\,\bar{c}'_R\,\bar{t}'_R)\, i\gamma^\mu D^{(uR)}_\mu \begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} + (\bar{d}'_R\,\bar{s}'_R\,\bar{b}'_R)\, i\gamma^\mu D^{(dR)}_\mu \begin{pmatrix} d'_R \\ s'_R \\ b'_R \end{pmatrix}$$

$$- (\bar{u}'_L\,\bar{c}'_L\,\bar{t}'_L)\, M_u \begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} - (\bar{d}'_L\,\bar{s}'_L\,\bar{b}'_L)\, M_d \begin{pmatrix} d'_R \\ s'_R \\ b'_R \end{pmatrix} + \text{H.c.}\,.$$

Note that $D^{(L)}_\mu$, $D^{(uR)}_\mu$, $D^{(dR)}_\mu$ are different, but they are diagonal in flavour space; and that we have re-introduced the convention of denoting „weak interaction eigenstates" with primed fields (cf. p. 26). Unprimed fields denote „mass eigenstates."

Diagonalization of mass matrices:

(i) Claim: The 3×3 complex matrices M_u, M_d can be diagonalized with „biunitary transformations":

$$\begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} = V_{uR} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \quad \begin{pmatrix} u'_L \\ c'_L \\ t'_L \end{pmatrix} = V_{uL} \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix}, \quad (d'_R) = V_{dR}(d_R),\ (d'_L) = V_{dL}(d_L)$$

so that

$$V^\dagger_{uL} M_u V_{uR} = \begin{pmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{pmatrix}, \quad V^\dagger_{dL} M_d V_{dR} = \begin{pmatrix} m_d & & 0 \\ & m_s & \\ 0 & & m_b \end{pmatrix}.$$

Here $m_u, m_c, m_t, m_d, m_s, m_b$ are real and positive; $V_{uR}, V_{uL}, V_{dR}, V_{dL}$ are unitary but not unique.

(ii) Construction: Consider the matrix $M_u M_u^\dagger$. This is Hermitean and therefore diagonalizable with real eigenvalues; in fact the eigenvalues are even non-negative (as reflected by $\det(M_u M_u^\dagger) = |\det M_u|^2 \geq 0$). So we can write

$$M_u M_u^\dagger = V_{uL} \begin{pmatrix} m_u^2 & & \\ & m_c^2 & \\ & & m_t^2 \end{pmatrix} V^\dagger_{uL}$$

$$= \underbrace{V_{uL}}_{\text{exists but not unique}} \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} V^\dagger_{uR} \underbrace{V_{uR}}_{\text{arbitrary at this point}} \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} V^\dagger_{uL}\,.$$

exists but not unique; each column could be multiplied by a phase, since

$$\begin{pmatrix} m_u^2 & & \\ & m_c^2 & \\ & & m_t^2 \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & & \\ & e^{i\beta} & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} m_u^2 & & \\ & m_c^2 & \\ & & m_t^2 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & & \\ & e^{-i\beta} & \\ & & e^{-i\gamma} \end{pmatrix}$$

Similarly, diagonalizing the Hermitean matrix $M_u^\dagger M_u$ we can determine a unitary matrix V_{uR} which again is fixed only up to phase transformations of the columns.

The same procedure can be repeated for V_{dR}, V_{dL}.

After the diagonalization, the mass terms read

$$\delta\mathcal{L} = -(\bar{u}_L\,\bar{c}_L\,\bar{t}_L) \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} - (\bar{d}_L\,\bar{s}_L\,\bar{b}_L) \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} + \text{H.c.}\,.$$

How do the kinetic terms look like in the new basis?

Neutral currents: These come from the „diagonal" kinetic terms, i.e.

(cf. p. 23)

$$\delta\mathcal{L} = (\bar{u}'_L\ \bar{c}'_L\ \bar{t}'_L)[i\gamma^\mu D^{(L)}_\mu]_{11}\begin{pmatrix}u'_L\\c'_L\\t'_L\end{pmatrix} + (\bar{d}'_L\ \bar{s}'_L\ \bar{b}'_L)[i\gamma^\mu D^{(L)}_\mu]_{22}\begin{pmatrix}d'_L\\s'_L\\b'_L\end{pmatrix}$$

$$+ (\bar{u}'_R\ \bar{c}'_R\ \bar{t}'_R)\, i\gamma^\mu D^{(uR)}_\mu \begin{pmatrix}u'_R\\c'_R\\t'_R\end{pmatrix} + (\bar{d}'_R\ \bar{s}'_R\ \bar{b}'_R)\, i\gamma^\mu D^{(dR)}_\mu \begin{pmatrix}d'_R\\s'_R\\b'_R\end{pmatrix}.$$

Clearly we can directly replace the weak interaction eigenstates by mass eigenstates, because $V^\dagger_{uL}V_{uL} = \mathbb{1}$ etc.

Charged currents:

(cf. p. 20)

$$\delta\mathcal{L} = (\bar{u}'_L\ \bar{c}'_L\ \bar{t}'_L)[i\gamma^\mu D^{(L)}_\mu]_{12}\begin{pmatrix}d'_L\\s'_L\\b'_L\end{pmatrix} + (\bar{d}'_L\ \bar{s}'_L\ \bar{b}'_L)[i\gamma^\mu D^{(L)}_\mu]_{21}\begin{pmatrix}u'_L\\c'_L\\t'_L\end{pmatrix}.$$

Now the two matrices do not multiply to unity:

$$V^\dagger_{uL}V_{dL} =: V_{CKM} =: \begin{pmatrix}V_{ud} & V_{us} & V_{ub}\\ V_{cd} & V_{cs} & V_{cb}\\ V_{td} & V_{ts} & V_{tb}\end{pmatrix},$$

Cabibbo 1963: 2x2 - matrix & charm quark
Kobayashi, Maskawa 1973: 3x3 - matrix & CP - violation

so that in the mass eigenbasis, charged current interactions are described by

$$\delta\mathcal{L} = (\bar{u}_L\ \bar{c}_L\ \bar{t}_L)[i\gamma^\mu D^{(L)}_\mu]_{12}\, V_{CKM}\begin{pmatrix}d_L\\s_L\\b_L\end{pmatrix} + \text{H.c.}\,.$$

Remarks:

(i) The fact that no matrix appears in neutral currents is referred to as the absence of „flavour changing neutral currents" (FCNC) at tree level. Many modifications of the Standard Model induce large FCNCs, and are therefore in conflict with experimental data.

(ii) How about leptons? Because neutrinos are very light, it may be a good approximation to set the corresponding matrix M_ν to zero. Then there would be no constraint on $V_{\nu L}$ from the diagonalization of the mass matrix; in particular, without changing physics, we could choose $V_{\nu L} := V_{eL}$. As a result

$$V^\dagger_{\nu L}V_{eL} = \mathbb{1},$$

so that there is no mixing.

If, in contrast, $M_\nu \neq 0$, there is mixing also in the lepton sector. The corresponding matrix is known as the PMNS - matrix, for Pontocorvo, Maki, Nakagawa and Sakata.
(Because of subtleties with neutrino masses, cf. sec 5.2, its structure may differ from that of V_{CKM}.)

How many physical parameters are there in V_{CKM}?

* A general $N\times N$ complex matrix has $2N^2$ real parameters.
  If the matrix is unitary, there are constraints to satisfy:
  from the diagonal part of $U^\dagger U = \mathbb{1}$, N real constraints;
  from the off-diagonal part, $2\times\frac{N(N-1)}{2} = N^2 - N$ real constraints.
  So, there are $2N^2 - N - (N^2-N) = N^2$ real parameters left over.

* However, V_{CKM} is not a most general unitary matrix:
  it is defined as a product of two unitary matrices (p.38), both of which contain N unphysical phases (p.37). So, we should subtract $2N$ real parameters.

* But now we subtracted a little bit too much: if all $2N$ phases are the same, $V_{CKM} = V_{uL}^\dagger V_{dL}$ does not change.
  So, the total number of physical parameters is

$$N^2 - 2N + 1 = \boxed{(N-1)^2} .$$

Check:

$N=1 \Rightarrow$ no mixing $\Rightarrow$ ok!

$N=2 \Rightarrow 1$ parameter $\Rightarrow$ Cabibbo angle (p.19).

$N=3 \Rightarrow 4$ parameters.

* Let us compare this with rotations, i.e. $O(N)$.
  An orthogonal matrix has $\frac{N(N-1)}{2}$ free parameters (Exercise 9.2(c)).

So:

$N=1 \Rightarrow 0 \Rightarrow$ ok!

$N=2 \Rightarrow 1 \Rightarrow$ Cabibbo angle.

$N=3 \Rightarrow 3$ (e.g. Euler angles).

* What this means is that V_{CKM} has one parameter which does not correspond to a rotation; it is a complex phase.
  A parametrization (à la Wolfenstein):

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

with $\lambda \approx 0.23$, $A \approx 0.8$, $\rho \approx 0.1$, $\eta \approx 0.3$.

* The existence of a complex phase implies that the combined symmetry CP is violated:

$$V_{ub}\,\bar{u}\,[i\gamma^\mu D_\mu]_{12}\,\frac{1-\gamma_5}{2}\,b \;\underset{(p.14)}{\xrightarrow{P}}\; V_{ub}\,\bar{u}\,[i\gamma^\mu D_\mu]_{12}\,\frac{1+\gamma_5}{2}\,b \;\underset{(p.15)}{\xrightarrow{C}}\; V_{ub}\,\bar{b}\,[i\gamma^\mu D_\mu]_{21}\,\frac{1-\gamma_5}{2}\,u$$

$$\neq V_{ub}^*\,\bar{b}\,[i\gamma^\mu D_\mu]_{21}\,\frac{1-\gamma_5}{2}\,u .$$

(This is the term arising from „H.c.")

Basics of CP-violation

CP-violation had been discovered already in 1964 (Cronin, Fitch ⇒ Nobel 1980), before Kobayashi and Maskawa explained it through the introduction of the 3rd generation and the corresponding phase in V_{CKM} (⇒ Nobel 2008).

Phenomenology:

(cf. Exercise 10.2)

$$|K^0_+\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \quad \text{with} \quad \hat{C}\hat{P}\,|K^0_+\rangle = +\,|K^0_+\rangle$$

$$\stackrel{\uparrow}{=} |K^0_S\rangle \text{ , i.e. „short-lived" mass eigenstate.}$$

If there is no CP-violation.

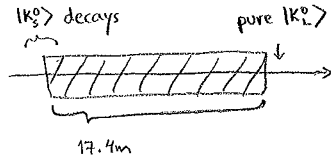
$$|K^0_-\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right) \quad \text{with} \quad \hat{C}\hat{P}\,|K^0_-\rangle = -\,|K^0_-\rangle$$

$$\stackrel{\downarrow}{=} |K^0_L\rangle \text{ , i.e. „long-lived" mass eigenstate.}$$

$$|K^0_S\rangle \to \overbrace{|\pi^+\rangle|\pi^-\rangle}^{CP=+1} \quad ; \quad \tau = 0.9 \times 10^{-10}\,\text{s} \quad ; \quad c\tau = 2.7\,\text{cm}$$

$$|K^0_L\rangle \to \underbrace{|\pi^0\rangle|\pi^+\rangle|\pi^-\rangle}_{CP=-1} \quad ; \quad \tau = 5.2 \times 10^{-8}\,\text{s} \quad ; \quad c\tau = 15.5\,\text{m}$$

$$|K^0\rangle \stackrel{\downarrow}{=} \frac{1}{\sqrt{2}}\left(|K^0_S\rangle + |K^0_L\rangle\right)$$



22700 decays; but with 45 $\pi^+\pi^-$ or $\pi^0\pi^0$!

So, CP is violated !

Generated through strong interactions as an eigenstate of strangeness, $|K^0\rangle |\bar{K}^0\rangle$; decays through weak interactions.

Significance:

It is believed that CP-violation is a necessary ingredient for explaining how a Universe with more matter than antimatter could emerge from a symmetric initial state. Thought experiment:

Initially: (matter) ⇒ ⇐ (antimatter)

After $\hat{P}$: (antimatter) ⇒ ⇐ (matter)

After $\hat{C}$: (matter) ⇒ ⇐ (antimatter)

If CP is violated, the final state differs from the initial state, i.e. something can be left over after pair annihilation.

Remark:

Strictly speaking, there could also be a second source of CP-violation in the Standard Model. This is related to the „anomalous" chiral rotation mentioned on p. 36, implicitly there also on p. 37. Some more words on this follow in sec. 4.

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