3.6 Scalars, fermions

We start by discussing how the „potential" $V(\Phi^\dagger\Phi)$ should be chosen in order to justify the crucial assumption made on p. 29. Subsequently we crosscheck that the same assumption gives masses to fermions as well.

P. 28: $\quad \delta\mathcal{L} = -V(\Phi^\dagger\Phi) \quad ; \quad V(\Phi^\dagger\Phi) = \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2$.

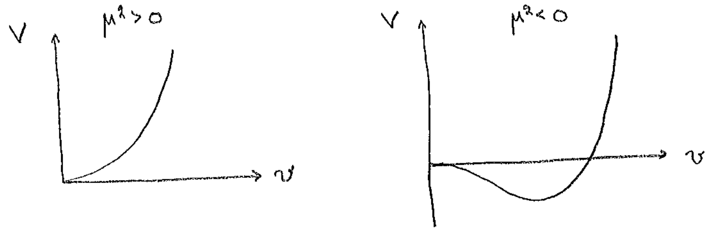
Recall also: $L = T - V$, $H = T + V$; that is, $V(\Phi^\dagger\Phi)$ represents energy density.

Basic argument:

Insert $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$ and inspect energy density as a function of v:

$$V = \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^4 .$$

We must choose $\lambda > 0$ in order to keep V bounded from below, but μ^2 could have either sign:



In the latter case $v > 0$ is favoured! So, this could justify our assumption.

Historically, the phenomenon is referred to as „Spontaneous symmetry breaking", although strictly speaking this is a correct term only for „global" symmetries (cf. sec. 4.1), not for „local" gauge symmetries.

Why do we speak of „symmetry breaking"? The reason is that $V(\Phi^\dagger\Phi)$ has the same value for $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$ or any gauge transformation thereof (cf. Exercise 7.1), e.g.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\iv\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}v\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}iv\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-v\end{pmatrix} .$$

Yet we choose $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$ as a representative. This is analogous to the magnetization direction of a ferromagnet.

Change of notation:

To underline the parameter choice made, we denote from now on

$$V(\Phi^\dagger\Phi) := -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 ,$$

where $\mu^2, \lambda > 0$.

Refined argument: Let us parametrize the Higgs field like on p. 31:

$$\Phi = \frac{1}{\sqrt{2}}\left[(v+\phi_0)\mathbb{1} + i\phi_a \tau^a\right]\begin{pmatrix}0\\1\end{pmatrix} \quad ;$$

$$\Phi^\dagger = \frac{1}{\sqrt{2}}(0\ 1)\left[(v+\phi_0)\mathbb{1} - i\phi_b \tau^b\right] \quad ;$$

$$\Phi^\dagger\Phi = \frac{1}{2}\left\{(v+\phi_0)^2 + \phi_a\phi_a\right\} .$$

We are looking for a minimum of $V(\Phi^\dagger\Phi)$, i.e.

(i) an extremum : there are no terms linear in ϕ_0, ϕ_a ($\partial_{\phi_\mu} V = 0$, $\mu \in \{0, a\}$).

(ii) with non-negative curvature : the matrix $\partial_{\phi_\mu}\partial_{\phi_\nu} V$ should not have negative eigenvalues.

Just carry out an explicit computation:

$$V(\Phi^\dagger\Phi) = -\frac{\mu^2}{2}\left[(v+\phi_0)^2 + \phi_a\phi_a\right] + \frac{\lambda}{4}\left[(v+\phi_0)^2 + \phi_a\phi_a\right]^2$$

$$= -\frac{\mu^2 v^2}{2} + \frac{\lambda v^4}{4} \qquad (a)$$

$$+\, v\phi_0\left(-\mu^2 + \lambda v^2\right) \qquad (b)$$

$$+\, \frac{\phi_0^2}{2}\left(-\mu^2 + 3\lambda v^2\right) + \frac{\phi_a\phi_a}{2}\left(-\mu^2 + \lambda v^2\right) \qquad (c)$$

$$+\, \lambda v \phi_0\left(\phi_0^2 + \phi_a\phi_a\right) + \frac{\lambda}{4}\left(\phi_0^2 + \phi_a\phi_a\right)^2 . \qquad (d)$$

Here:

(a) → constant (no ϕ_0, ϕ_a)

(b) → must vanish according to (i)

$\Rightarrow \quad v = 0 \quad \vee \quad v^2 = \frac{\mu^2}{\lambda}$.

(c) → negative masses for $v = 0$ ↯ .
But for $v^2 = \frac{\mu^2}{\lambda}$, we get

$$\frac{\phi_0^2}{2}\times(2\mu^2) + \frac{\phi_a\phi_a}{2}\times(0) \qquad \text{ok!}$$

(d) → „interactions" .

So a desired minimum exists at $\boxed{v^2 = \frac{\mu^2}{\lambda}.}$

Summary:

* The fluctuation ϕ_0, called the „Higgs boson", has a gauge-independent mass $M_H^2 = 2\mu^2 = 2\lambda v^2 > 0$.

* The fluctuations ϕ_a, called „Goldstone modes", got no mass from here but did get a mass from gauge fixing, cf. p. 32. We say that the Goldstone modes are „eaten up" by the vector bosons and do not appear as independent physical degrees of freedom (cf. Exercise 9.1).

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Fermion masses:

We expect massive fermions to be described by the same type of Lagrangian as in QED (p. 5):

$$\mathcal{L} = \sum_f \bar{\Psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \Psi_f \; .$$

Here (cf. Exercise 4.3):

$$\bar{\Psi}_f \, i\gamma^\mu D_\mu \Psi_f = \bar{\Psi}_f P_R i\gamma^\mu D_\mu P_L \Psi_f + \bar{\Psi}_f P_L i\gamma^\mu D_\mu P_R \Psi_f$$

$$= \bar{\Psi}_{fL} \, i\gamma^\mu D_\mu \Psi_{fL} + \bar{\Psi}_{fR} \, i\gamma^\mu D_\mu \Psi_{fR}$$

$$\boxed{\mathbb{1} = P_L + P_R = P_L^2 + P_R^2}$$

$$-\bar{\Psi}_f m_f \Psi_f = -\bar{\Psi}_f P_L \, m_f P_L \Psi_f - \bar{\Psi}_f P_R \, m_f P_R \Psi_f$$

$$= -\bar{\Psi}_{fR} \, m_f \Psi_{fL} - \bar{\Psi}_{fL} \, m_f \Psi_{fR} \; .$$

The starting point now (p. 28; Exercise 7.3):

$$\delta\mathcal{L} = -\Big[h_u^{ij} \bar{Q}_{iL} \tilde{\Phi} u_{jR} + h_d^{ij} \bar{Q}_{iL} \Phi d_{jR} + h_\nu^{ij} \bar{L}_{iL} \tilde{\Phi} \nu_{jR} + h_e^{ij} \bar{L}_{iL} \Phi e_{jR} + \text{H.c.} \Big] ,$$

where (p. 28; Exercise 7.2)

$$\tilde{\Phi} = i\tau^2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix} .$$

Basic argument:

Let us start by considering <u>one generation</u> $(i,j \to 1)$ and <u>real couplings</u> $(h_u^{11}, h_d^{11}, h_\nu^{11}, h_e^{11} \in \mathbb{R})$.

For the „<u>down-type</u>" fermions:

$$\Phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\bar{Q}_{1L} \Phi \to \frac{1}{\sqrt{2}} \bar{d}_{1L} v$$

$$\delta\mathcal{L} \to -\frac{h_d^{11} v}{\sqrt{2}} \left[\bar{d}_{1L} d_{1R} + \text{H.c.} \right]$$

$$=: -m_d \left[\bar{d}_L d_R + \bar{d}_R d_L \right] ,$$

with $m_d = \dfrac{h_d^{11} v}{\sqrt{2}}$.

For the „<u>up-type</u>" fermions:

$$\tilde{\Phi} \to \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$\delta\mathcal{L} \to -\frac{h_u^{11} v}{\sqrt{2}} \left[\bar{u}_{1L} u_{1R} + \text{H.c.} \right]$$

$$=: -m_u \left[\bar{u}_L u_R + \bar{u}_R u_L \right] ,$$

with $m_u = \dfrac{h_u^{11} v}{\sqrt{2}}$.

Electrons obtain a mass just like down-type quarks.

(We could also get masses for neutrinos, although in that case the situation turns out to be a little bit more complicated; we return to this in sec. 5.2.)

Refined argument:

(a) What if the couplings are complex?

For instance: $\quad h_u^{11} =: h_u = |h_u| e^{i\alpha}$;

$$\delta \mathcal{L}_u = \bar{u}_L i\gamma^\mu D_\mu u_L + \bar{u}_R i\gamma^\mu D_\mu u_R - \frac{|h_u| v}{\sqrt{2}} \left(\bar{u}_L u_R e^{i\alpha} + \bar{u}_R u_L e^{-i\alpha} \right).$$

Now, on the „classical level", we can make the following „chiral rotation":

$$u_R \to e^{-\frac{i\alpha}{2}} u_R \quad ; \quad \bar{u}_R \to \bar{u}_R e^{\frac{i\alpha}{2}} \; ;$$

$$u_L \to e^{\frac{i\alpha}{2}} u_L \quad ; \quad \bar{u}_L \to \bar{u}_L e^{-\frac{i\alpha}{2}} .$$

Or:

$$u \to u' = e^{-\frac{i\alpha\gamma_5}{2}} u = \Big(1 - \frac{i\alpha\gamma_5}{2} + \frac{1}{2}\Big(\frac{-i\alpha}{2}\Big)^2 \overbrace{\gamma_5^2}^{1} + \ldots\Big) u$$

$$= \cos\Big(\frac{\alpha}{2}\Big) u - i\sin\Big(\frac{\alpha}{2}\Big)\gamma_5 u \; ;$$

$$u_R \to u_R' = \cos\Big(\frac{\alpha}{2}\Big) u_R - i\sin\Big(\frac{\alpha}{2}\Big) P_R \gamma_5 u$$

$$= \cos\Big(\frac{\alpha}{2}\Big) u_R - i\sin\Big(\frac{\alpha}{2}\Big) P_R u = e^{-\frac{i\alpha}{2}} u_R \; ;$$

$$u_L \to u_L' = \cos\Big(\frac{\alpha}{2}\Big) u_L - i\sin\Big(\frac{\alpha}{2}\Big) \underbrace{P_L \gamma_5}_{-P_L} u$$

$$= \cos\Big(\frac{\alpha}{2}\Big) u_L + i\sin\Big(\frac{\alpha}{2}\Big) P_L u = e^{\frac{i\alpha}{2}} u_L .$$

Clearly this transformation leaves the kinetic terms invariant and removes the phase from the mass term.

(It turns out that on the „quantum level" this transformation is quite subtle, because the path integration measure is not necessarily invariant; we speak of a „quantum anomaly", cf. sec. 4.2.)

(b) What if the couplings have non-diagonal components?

Let us introduce 3x3-matrices as

$$(M_u)_{ij} := \frac{h_u^{ij} v}{\sqrt{2}} \quad , \quad \text{etc.}$$

Then the mass terms have the structure

$$\delta\mathcal{L} = -\Bigg[(\bar{u}_L\ \bar{c}_L\ \bar{t}_L) M_u \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + (\bar{u}_R\ \bar{c}_R\ \bar{t}_R) M_u^\dagger \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix}$$

$$+ (\bar{d}_L\ \bar{s}_L\ \bar{b}_L) M_d \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} + (\bar{d}_R\ \bar{s}_R\ \bar{b}_R) M_d^\dagger \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

$$+ \text{leptons} \Bigg] .$$

We return to the analysis of these terms in sec. 3.7.

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