3.5 Vector bosons

The goal is to crosscheck that the Standard Model does give the intermediate vector bosons masses as required by sec. 3.3. Starting point:

$$\mathcal{L} = \ldots + (D_\mu \Phi)^\dagger (D^\mu \Phi) \; ; \qquad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \; ;$$

p. 27

$$D_\mu = \partial_\mu - i g_w T^a A^a_\mu + i g_Y T^0 B_\mu \; ; \quad {}^*$$

$$T^a = \frac{\tau^a}{2} \; ; \quad T^0 = \frac{\mathbb{1}_{2\times 2}}{2} \, .$$

Note: Like leptons, the Higgs is „charged" under $U_Y(1)$ and $SU_L(2)$ but not $SU_c(3)$. Sometimes one speaks of a „unified" theory of electromagnetic and weak interactions.

Basic argument: Let us assume that the dynamics of the Higgs field is such that Φ can be approximated as a constant:**

A „dynamical" realization of this assumption is discussed in sec. 3.6.

** The precise form involves a convention, cf. Exercise 8.2.

$$\Phi \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \; , \qquad v \in \mathbb{R} \, .$$

Then

$$D_\mu \Phi = \frac{-i}{\sqrt{2}} \left(g_w T^b A^b_\mu - g_Y T^0 B_\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix} \; ;$$

$$(D_\mu \Phi)^\dagger = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & v \end{pmatrix} \left(g_w T^a A^a_\mu - g_Y T^0 B_\mu \right)$$

symmetric in $a \leftrightarrow b$

$$\Rightarrow \quad \delta\mathcal{L} = \frac{v^2}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \Big\{ g_w^2 \, T^a T^b A^{a\mu} A^b_\mu - 2 g_w g_Y \, T^a T^0 A^{a\mu} B_\mu + g_Y^2 \, T^0 T^0 B^\mu B_\mu \Big\} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, .$$

Here

$$T^a T^b = \frac{1}{4} \tau^a \tau^b = \frac{1}{4} \left\{ \delta^{ab} \mathbb{1} + i \epsilon^{abc} \tau^c \right\} \; ;$$

antisymmetric in $a \leftrightarrow b$

$$T^a T^0 = \frac{1}{4} \tau^a \; ; \qquad (\tau^a)_{22} = - \delta_{a3}$$

$$T^0 T^0 = \frac{1}{4} \mathbb{1}$$

$$\Rightarrow \quad \delta\mathcal{L} = \frac{1}{2} \left(\frac{v}{2} \right)^2 \Big\{ g_w^2 \sum_{a=1}^{2} A^{a\mu} A^a_\mu + g_w^2 A^{3\mu} A^3_\mu + 2 g_w g_Y A^{3\mu} B_\mu + g_Y^2 B^\mu B_\mu \Big\}$$

$$= \frac{1}{2} \Bigg\{ \left(\frac{g_w v}{2} \right)^2 \sum_{a=1}^{2} A^{a\mu} A^a_\mu + \left(\frac{\sqrt{g_w^2 + g_Y^2}\, v}{2} \right)^2 \left(\frac{g_w}{\sqrt{g_w^2+g_Y^2}} A^{3\mu} + \frac{g_Y}{\sqrt{g_w^2+g_Y^2}} B^\mu \right) \times \left(\frac{g_w}{\sqrt{g_w^2+g_Y^2}} A^3_\mu + \frac{g_Y}{\sqrt{g_w^2+g_Y^2}} B_\mu \right) \Bigg\} \, .$$

We realize that $\left| \frac{g_w}{\sqrt{g_w^2+g_Y^2}} \right|, \left| \frac{g_Y}{\sqrt{g_w^2+g_Y^2}} \right| < 1 \, .$

---

* Since this is the first time that g_Y is introduced, the hypercharge assignment „$Y = -\frac{1}{2}$" for Φ can be considered a convention.

Then we define new linear combinations of fields, in analogy with the Cabibbo rotation (p. 19):

$$\begin{pmatrix} Z_\mu \\ Q_\mu \end{pmatrix} := \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} ,$$

where θ_W = „Weinberg - angle" $\approx 28.7°$ *.

* θ_W is also known as the „weak mixing angle".

Identifying

$$\cos\theta_W := \frac{g_W}{\sqrt{g_W^2 + g_Y^2}} , \qquad \sin\theta_W := \frac{g_Y}{\sqrt{g_W^2 + g_Y^2}} ,$$

$$M_W := \frac{g_W v}{2} , \qquad M_Z := \frac{\sqrt{g_W^2 + g_Y^2}\, v}{2} ,$$

the Lagrangian can be written as

$$\delta\mathcal{L} = \frac{1}{2} \left\{ M_W^2 \sum_{a=1}^{2} A^{a\mu} A^a_\mu + M_Z^2 Z^\mu Z_\mu + 0 \cdot Q^\mu Q_\mu \right\} .$$

So, comparing with p. 25, this Lagrangian describes two vector bosons with a mass M_W ($\Rightarrow W^\pm$); one vector boson with a mass $M_Z > M_W$ ($\Rightarrow Z^0$); and one massless vector boson ($\Rightarrow \gamma$).

<u>Couplings of the Higgs:</u>

Having rotated the fields, let us rewrite the covariant derivative in terms of the new ones:

$$D_\mu = \partial_\mu - \frac{i g_W}{2} \begin{pmatrix} 0 & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu & 0 \end{pmatrix} \underbrace{- \frac{i g_W}{2} \begin{pmatrix} A^3_\mu & 0 \\ 0 & -A^3_\mu \end{pmatrix} + \frac{i g_Y}{2} \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix}}_{}$$

$$\frac{i\sqrt{g_W^2 + g_Y^2}}{2} \begin{pmatrix} -\cos\theta_W A^3_\mu + \sin\theta_W B_\mu & 0 \\ 0 & \cos\theta_W A^3_\mu + \sin\theta_W B_\mu \end{pmatrix}$$

$$\boxed{\begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ Q_\mu \end{pmatrix}}$$

$$= \frac{i\sqrt{g_W^2 + g_Y^2}}{2} \begin{pmatrix} [-\cos^2\theta_W + \sin^2\theta_W] Z_\mu + [2\cos\theta_W \sin\theta_W] Q_\mu & 0 \\ 0 & Z_\mu \end{pmatrix}$$

So:

* $A^1_\mu \mp i A^2_\mu$ are in the off-diagonal components and should therefore couple to charged currents, as expected from $W^\pm$.
* ϕ^0 does <u>not</u> couple to Q_μ; it is electrically neutral, as suggested by the notation.
* ϕ^+ does couple to Q_μ; if we denote the coupling by e like on p. 5, we get the relation

$$e = \sqrt{g_W^2 + g_Y^2}\, \cos\theta_W \sin\theta_W$$

$$\Leftrightarrow \boxed{e = g_W \sin\theta_W} .$$

Refined argument: The Higgs field cannot really be exactly constant. However we might assume that typical fluctuations are small compared with some average value:

$$\Phi =: \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_1 \\ v + \phi_0 - i\phi_3 \end{pmatrix} \overset{!}{=} \frac{1}{\sqrt{2}} \left[(v+\phi_0)\mathbb{1} + i\phi_a \tau^a \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $v, \phi_0, \phi_a \in \mathbb{R}$, and $|\phi_0|, |\phi_a| \ll v$ in some sense.

Then we need to consider the quadratic part of the Lagrangian, in order to determine the propagators and thereby the („tree-level") masses.

Now:
$$D_\mu \Phi = \frac{1}{\sqrt{2}} \left[\partial_\mu \phi_0 \mathbb{1} + i \partial_\mu \phi_a \tau^a \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{i}{\sqrt{2}} \left[g_w T^a A^a_\mu - g_Y T^0 B_\mu \right] \begin{pmatrix} 0 \\ v \end{pmatrix}$$
$$+ \text{(quadratic terms)}$$

$$(D_\mu \Phi)^\dagger = \frac{(0\,1)}{\sqrt{2}} \left[\partial_\mu \phi_0 \mathbb{1} - i \partial_\mu \phi_b \tau^b \right] + \frac{i}{\sqrt{2}} (0\, v) \left[g_w T^b A^b_\mu - g_Y T^0 B_\mu \right]$$
$$+ \text{(quadratic terms)}.$$

There are three structures emerging from here:

(i) The terms that we already had on p. 29.

(ii) Kinetic terms for the fluctuations:

$$\delta \mathcal{L} = \frac{1}{2} (0\,1) \left[\partial_\mu \phi_0 \mathbb{1} - i \partial_\mu \phi_b \tau^b \right] \left[\partial^\mu \phi_0 \mathbb{1} + i \partial^\mu \phi_a \tau^a \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} (0\,1) \left[\partial_\mu \phi_0 \partial^\mu \phi_0 \mathbb{1} + \underbrace{\partial_\mu \phi_b \partial^\mu \phi_a}_{\text{symmetric in } a \leftrightarrow b} \tau^b \tau^a \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\tau^a \tau^b = \delta^{ab} \mathbb{1} + i \epsilon^{abc} \tau^c$ — antisymmetric

$$= \frac{1}{2} \left[\partial_\mu \phi_0 \partial^\mu \phi_0 + \partial_\mu \phi_a \partial^\mu \phi_a \right]$$

(iii) „Mixed terms":

$$\delta \mathcal{L} = \frac{(0\,1)}{4} \left[-i \partial_\mu \phi_0 \mathbb{1} - \partial_\mu \phi_b \tau^b \right] \left[g_w A^{a\mu} \tau^a - g_Y B^\mu \mathbb{1} \right] \begin{pmatrix} 0 \\ v \end{pmatrix}$$
$$+ \frac{(0\,v)}{4} \left[g_w A^{a\mu} \tau^a - g_Y B^\mu \mathbb{1} \right] \left[i \partial_\mu \phi_0 \mathbb{1} - \partial_\mu \phi_b \tau^b \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

part with $\partial_\mu \phi_0$ drops out

$$= \frac{(0\,1)}{4} \left[-\partial_\mu \phi_b \, g_w A^{a\mu} \{\tau^b, \tau^a\} + \partial_\mu \phi_b \, g_Y B^\mu \{\tau^b, \mathbb{1}\} \right] \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$= \frac{v}{2} \left[-g_w \partial_\mu \phi_a A^{a\mu} - g_Y \partial_\mu \phi_3 B^\mu \right].$$

So gauge fields and scalars appear to become „intertwined".

Gauge fixing:

A way to remove the mixing is to recall that the Higgs doublet Φ is not gauge-invariant, and determining the propagators of gauge fields requires carrying out a suitable gauge fixing.

In QED we had (Exercise 2.1):

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 .$$

Now we are free to make another gauge choice, which also involves the gauge non-invariant scalar fields. A particularly convenient choice is called the R_ξ - gauge:

$$\mathcal{L}_{g.f.} := -\frac{1}{2\xi_W}\sum_{a=1}^{3}(G_W^a)^2 - \frac{1}{2\xi_Y}G_Y^2 ;$$

$$G_W^a := \partial_\mu A^{a\mu} + \xi_W \frac{g_W v}{2}\phi_a ;$$

$$G_Y := \partial_\mu B^\mu + \xi_Y \frac{g_Y v}{2}\phi_3 .$$

$$\Rightarrow \quad \mathcal{L}_{g.f.} = -\frac{1}{2\xi_W}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) - \frac{1}{2\xi_Y}(\partial_\mu B^\mu)(\partial_\nu B^\nu)$$
$$- \frac{g_W v}{2}\partial_\mu A^{a\mu}\phi_a - \frac{g_Y v}{2}\partial_\mu B^\mu \phi_3$$
$$- \frac{1}{2}\xi_W\left(\frac{g_W v}{2}\right)^2\phi_a\phi_a - \frac{1}{2}\xi_Y\left(\frac{g_Y v}{2}\right)^2\phi_3\phi_3 .$$

Combining with p. 31 the mixed terms become

$$\delta\mathcal{L} = -\frac{g_W v}{2}\left(A^{a\mu}\partial_\mu\phi_a + \partial_\mu A^{a\mu}\phi_a\right)$$
$$-\frac{g_Y v}{2}\left(B^\mu\partial_\mu\phi_3 + \partial_\mu B^\mu\phi_3\right)$$
$$= -\frac{g_W v}{2}\partial_\mu(A^{a\mu}\phi_a) - \frac{g_Y v}{2}\partial_\mu(B^\mu\phi_3) .$$

However total derivatives play no role in the Lagrangian formalism [or in a path integral with periodic boundary conditions].

Lesson:

The analysis of p. 29-30 is OK for the gauge fields. At the same time, the scalar field components ϕ_a, $a=1,2,3$, are seen to obtain gauge-dependent masses (cf. the last line of $\mathcal{L}_{g.f.}$). So these fields cannot represent physical degrees of freedom $\Rightarrow$ sec. 3.6.