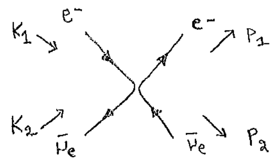
3.3 Problems of the Fermi model

The V-A Fermi model works very well at low energies ($s \ll G_F^{-1}$), but potential problems arise* if we try to compute (i) high-energy cross sections ($s \gg G_F^{-1}$) or (ii) radiative corrections from virtual particles.

(i) Violation of the "unitarity bound"

As an example consider

$\bar{\nu}_e e^- \to \bar{\nu}_e e^-$.



$s = (K_1+K_2)^2 = (P_1+P_2)^2$

How does σ behave for large s?

Dimensional analysis:

* $\mathcal{M}$ has always G_F

 $\Rightarrow |\mathcal{M}|^2$ — " — G_F^2

* $[\sigma] = [\text{area}] = [\text{fm}^2] = \frac{1}{\text{GeV}^2}$; $[G_F] = \frac{1}{\text{GeV}^2}$

 $\Rightarrow \sigma \propto G_F^2 s$.

* The exact answer: $\sigma = \frac{G_F^2 s}{3\pi}$.

However this cannot be correct: there is a universal bound, called the "Froissart bound", which states that σ can grow at most logarithmically at large s.

The bound is related to the "conservation of probability" familiar from quantum mechanics, requiring that the Hamiltonian be hermitean and the time evolution operator be unitary:

$$\underbrace{\hat{S}}_{\text{"S-matrix"}} := \hat{U}_I(\infty,-\infty) =: \underbrace{\hat{\mathbb{1}}}_{\text{"nothing happens"}} - i \underbrace{\hat{T}}_{\text{"transfer matrix", describing scatterings}}$$

$$\hat{S}^+\hat{S} = \hat{\mathbb{1}}$$

$$\Leftrightarrow (\hat{\mathbb{1}} + i\hat{T}^+)(\hat{\mathbb{1}} - i\hat{T}) = \hat{\mathbb{1}}$$

$$\Leftrightarrow \boxed{i(\hat{T} - \hat{T}^+) = \hat{T}^+\hat{T}} .$$

This relation is valid for any scattering and imposes non-linear relations on how the matrix elements $\mathcal{M}_{fi} \sim \langle f|\hat{T}|i\rangle$ can grow.

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* Actually the arguments presented are not necessarily "full-proof", cf. Exercise 6.3. Nevertheless they have been very inspiring and led to the discovery of important new physics.

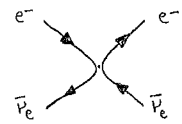
Solution to (i) [O. Klein 1938]

(a) Write $G_F =: \frac{g_W^2}{4\sqrt{2}\, M_W^2}$, where g_W, M_W are so far unknown parameters.

(b) Denote $J_+^\mu := \sum_{D_L} \bar{D}_L \begin{pmatrix} 0 & 0 \\ \gamma^\mu & 0 \end{pmatrix} D_L$; $J_-^\mu := (J_+^\mu)^\dagger = \sum_{D_L} \bar{D}_L \begin{pmatrix} 0 & \gamma^\mu \\ 0 & 0 \end{pmatrix} D_L$.

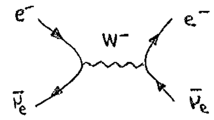
(c) With (a) and (b) the V-A Fermi model can be expressed as

$$\mathcal{L}_I^{V-A} = -\frac{g_W^2}{2} J_+^\mu \left(\frac{\eta_{\mu\nu}}{M_W^2} \right) J_-^\nu$$



(d) Now we modify the theory as follows:

$$\mathcal{L}_I^{V-A} \Rightarrow i \frac{g_W^2}{2} J_+^\mu \underbrace{\left(\frac{-i\eta_{\mu\nu}}{s - M_W^2 + i0^+} \right)}_{\text{like a photon propagator but with a mass !}} J_-^\nu$$



(e) As a consequence: $*$ for $s \ll M_W^2$ nothing changes ;

$*$ for $s \gg M_W^2$ we get

$$\sigma = \frac{G_F^2 s}{3\pi} \left(\frac{M_W^2}{s - M_W^2} \right)^2 \approx \frac{G_F^2 M_W^4}{3\pi s} \; !$$

So, through the introduction of a new vertex (similar to those in QED) and a new particle, the theoretical problem of unitarity violation has been solved.

The new particle W^- (and a corresponding W^+ for $e^+\nu_e \to e^+\nu_e$) is electrically charged. Therefore J_+^μ and J_-^μ are called <u>charged currents</u>.

Experimentally $W^\pm$ were discovered only in 1983 at CERN. Their mass is measured to be $M_W = (80.379 \pm 0.012)\,\text{GeV}$. They are also known as <u>intermediate vector bosons</u>.

Further developments

Although the introduction of $W^\pm$ solves the problem with $\bar\nu_e e^- \to \bar\nu_e e^-$, there are other processes where a problem remains. A full solution requires the introduction of <u>neutral currents</u> as well as a <u>neutral intermediate vector boson</u>, the Z^0 [Bludman 1958].

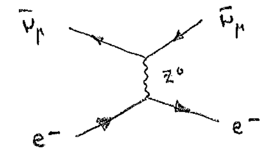
So : $*$ $$J_0^\mu := \sum_D \bar{D} \begin{pmatrix} c_V^u \gamma^\mu + c_A^u \gamma^\mu \gamma^5 & 0 \\ 0 & c_V^d \gamma^\mu + c_A^d \gamma^\mu \gamma^5 \end{pmatrix} D .$$

This is no longer just „V-A"; the numerical coefficients $c_V^u, c_A^u, c_V^d, c_A^d$ will be fixed later on. (they depend on D!).

$*$ $$\delta \mathcal{L}_I := i \, \frac{g_w^2}{2} \, J_0^\mu \left(\frac{-i\eta_{\mu\nu}}{s - M_Z^2} \right) J_0^\nu \; ; \quad M_Z = (91.1876 \pm 0.0021)\,\text{GeV}.$$

Consequences from neutral currents:

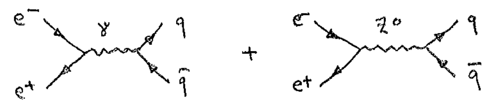
(a) Because c_V^u, c_A^u can be non-zero, neutrinos feel direct interactions, e.g. in $\bar\nu_\mu + e^- \to \bar\nu_\mu + e^-$:



„t-channel"

Neutral current processes were experimentally first studied through this channel [CERN 1973].

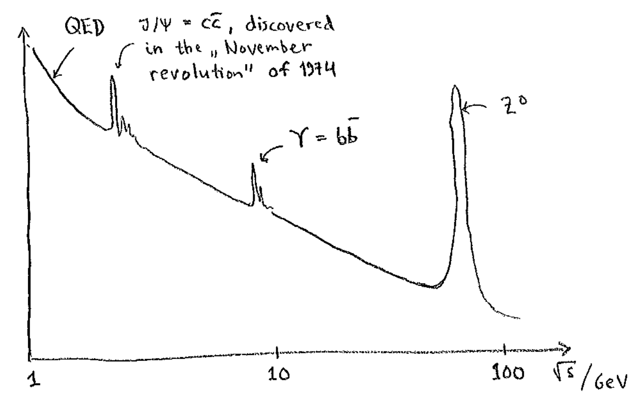
(b) Consider $e^- e^+ \to$ hadrons (cf. p. 12)



Exercise 3.4 $\Rightarrow \sigma \sim \frac{1}{s}$

Like p. 22: $\sigma \sim \frac{s}{(s - M_Z^2)^2}$

Experimentally:



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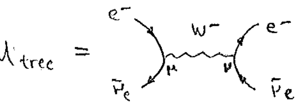
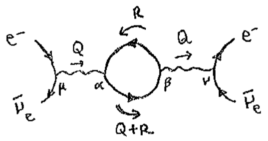
(ii) Problem of „infinitely large loop corrections"

Consider a „loop correction", e.g.

$\mathcal{M}_{tree} =$  ; $\mathcal{M}_{loop} =$ 

The „effective" inner line of the latter case:

$$\left(\frac{-i\eta_{\mu\alpha}}{Q^2-M_W^2}\right)\underbrace{\int\frac{d^4R}{(2\pi)^4}\,\mathrm{Tr}\left\{(ig_w\gamma^\alpha)\frac{i(\not R+m)}{R^2-m^2+i0^+}(ig_w\gamma^\beta)\frac{i(\not Q+\not R+m)}{(Q+R)^2-m^2+i0^+}\right\}}\left(\frac{-i\eta_{\nu\beta}}{Q^2-M_W^2}\right)$$

like p.10: $$4g_w^2\,\frac{R^\alpha(Q+R)^\beta+R^\beta(Q+R)^\alpha-\eta^{\alpha\beta}\left[R\cdot(Q+R)-m^2\right]}{\left[R^2-m^2+i0^+\right]\left[(Q+R)^2-m^2+i0^+\right]}$$

Let us inspect the contribution from the last term:

$$\begin{aligned}R\cdot(Q+R)-m^2 &= R^2+\tfrac{1}{2}\left[(Q+R)^2-Q^2-R^2\right]-m^2\\ &= \tfrac{1}{2}\left[R^2-m^2\right]+\tfrac{1}{2}\left[(Q+R)^2-m^2\right]-\tfrac{1}{2}Q^2\end{aligned}$$

$$\Rightarrow\ \frac{4g_w^2\,\eta^{\mu\nu}}{(Q^2-M_W^2)^2}\left\{\int\frac{d^4R}{(2\pi)^4}\frac{1}{R^2-m^2+i0^+}-\frac{Q^2}{2}\int\frac{d^4R}{(2\pi)^4}\frac{1}{\left[R^2-m^2+i0^+\right]\left[(Q+R)^2-m^2+i0^+\right]}\right\}$$

$\downarrow$ quadratically divergent! (cf. Exercise 6.2)

$\downarrow$ logarithmically divergent! (cf. Exercise 6.2)

Solution to (ii) ['t Hooft, Veltman 1971]

If the divergences can be „hidden" by computing relations between physical quantities, rather than physical quantities in terms of abstract „bare" parameters, then the theory is OK. Such a theory is called „renormalizable".

Theories can be shown to be renormalizable if:

(a) They are „gauge-invariant", so that many quantities are automatically related to each other;

(b) In the Lagrangian there are only vertices containing three or four bosons, or one boson and two fermions. This guarantees that the corresponding couplings have a non-negative dimensionality in GeV, and excludes also a powerlike unitarity violation like on p. 21.