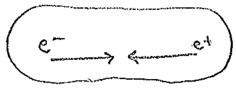
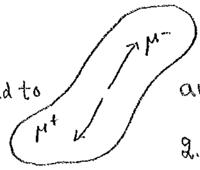
3. Weak interactions

3.1 Symmetries and conservation laws

According to QED, 

can lead to 

and $\mu^\pm$ live on forever.

But this is wrong: the muons decay after $2.2 \times 10^{-6}\,s$ (outside the detector).
Even though weak interactions are „weak", they are thus very important, because they lead to decays which would otherwise be forbidden. This is related to the fact that weak interactions break many symmetries.

Classification of symmetries: $\left\{\begin{array}{l}\text{space-time}\\ \text{inner (or internal)}\end{array}\right. \times \left\{\begin{array}{l}\text{continuous}\\ \text{discrete}\end{array}\right. \times \left\{\begin{array}{l}\text{global}\\ \text{local}\end{array}\right. \leftarrow$ later on!

Space-time symmetries: (i) Translations $\boxed{x^\mu \to x'^\mu = x^\mu + c^\mu}$.

In vacuum this is believed to be an unbroken symmetry.* The corresponding „Noether current" is the energy-momentum tensor; it represents a conserved quantity.

* The symmetry can be „spontaneously" broken in a multiparticle system, leading to the formation of a crystal.

(ii) Lorentz transformations $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$.

Invariance: $x'\cdot y' = \eta_{\mu\nu} x'^\mu y'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha y^\beta \stackrel{!}{=} \eta_{\alpha\beta} x^\alpha y^\beta$

$\Rightarrow \boxed{\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$.

(a) Choose $\alpha = \beta = 0$

$\Rightarrow (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1$

$\Rightarrow (\Lambda^0{}_0)^2 \geq 1$

$\Rightarrow \underbrace{\Lambda^0{}_0 \geq 1}_{\text{„orthochronous"}}$ or $\Lambda^0{}_0 \leq -1$.

„orthochronous" ; „$L^\uparrow$"

(b) Rewrite in matrix notation

$\Rightarrow \Lambda^T \eta \Lambda = \eta \quad | \det$

$\Rightarrow [\det(\Lambda)]^2 = 1$

$\Rightarrow \det(\Lambda) = +1$ or $\det(\Lambda) = -1$

„proper" ; „L_+"

The „subgroup" $L_+^\uparrow$ of proper orthochronous Lorentz transformations is believed to be an unbroken symmetry.*

* But again, the symmetry can be broken in a multiparticle system, in which the „average" flow velocity defines a preferred frame.

Without proof: let $\psi, \bar\psi$ transform under a specific „representation" of $L_+^\uparrow$:

$$\psi(x) \to \psi'(x') = S\,\psi(x)$$
$$\bar\psi(x) \to \bar\psi'(x') = \bar\psi(x)\,\underbrace{S^{-1}}_{4\times 4\text{-Matrix}}$$

Then $V^\mu := \bar\psi \gamma^\mu \psi$ and $A^\mu := \bar\psi\gamma^\mu\gamma^5\psi$, $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$,
transform as vectors: $V^{\mu\prime} = \Lambda^\mu{}_\nu V^\nu$, $A^{\mu\prime} = \underbrace{\det(\Lambda)}_{+1}\Lambda^\mu{}_\nu A^\nu$.

(iii) Parity $\boxed{x \to x' = \Lambda_P x \quad , \quad \Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}$

Parity is an „improper" Lorentz transformation : $\Lambda_P \in L_-$.

(In fact any $\Lambda \in L_-$ can be written as $\Lambda = \Lambda_P \Lambda'$, with $\Lambda' \in L_+$.
Proof: $\Lambda = \Lambda_P \underbrace{\Lambda_P \Lambda}_{\Lambda'}$; $\det(\Lambda') = \det(\Lambda_P)\det(\Lambda) = (-1)^2$ □ .)

Consider representations of parity in field space: $\begin{cases} \psi'(x') = P\psi(x) . \\ \bar\psi'(x') = \bar\psi(x) P^{-1} . \end{cases}$

(a) Since $\Lambda_P^2 = 1\!\!1$, we expect that $\psi''(x'') = \psi(x)$. So,

$$\psi''(x'') = \psi''(\Lambda_P \Lambda_P x) = P\psi'(\Lambda_P x) = P^2 \psi(x) \overset{!}{=} \psi(x)$$

i.e. $P^2 = 1\!\!1$.

(b) What happens with the action? Look at a free electron:

$$S'_\psi = \int_{x'} \bar\psi'(x') \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m \right) \psi'(x')$$

$$= \int_x \bar\psi(x) P^{-1} \left(i\gamma^0 \frac{\partial}{\partial x^0} - i\gamma^k \frac{\partial}{\partial x^k} - m \right) P \psi(x) .$$

$d^4x' = \underbrace{|\det \Lambda_P|}_{1} d^4 x$

We see that the action is invariant (a symmetry exists) if

$$\begin{cases} P^{-1}\gamma^0 P = \gamma^0 \\ P^{-1}\gamma^k P = -\gamma^k , \quad k = 1,2,3 . \end{cases}$$

All requirements can be satisfied with the choice $P = \gamma^0$.
So, QED is symmetric under parity.

Remarks:

* Not every action is symmetric under parity!
In particular:

$$S'_5 := \int_{x'} \bar\psi'(x') \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m \right) \gamma^5 \psi'(x')$$

$$= \int_x \bar\psi(x) \gamma^0 \left(i\gamma^0 \frac{\partial}{\partial x^0} - i\gamma^k \frac{\partial}{\partial x^k} - m \right) \gamma^5 \gamma^0 \psi(x)$$

$$\overset{\gamma^5\gamma^0 = -\gamma^0\gamma^5}{=} - S_5 .$$

* Similar considerations could be carried out with time reversal ; $x \to x' = \Lambda_T x$, $\Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.

We will not do this, since it turns out [W. Pauli 1955] that if the theory is $L_+^\uparrow$ - invariant and Hermitean, then T is equivalent to CP. Here C is „charge conjugation", to which we now turn.

Inner symmetries: (i) Charge conjugation

Roughly, we try to exchange $\psi \leftrightarrow \psi^*$, or particle $\leftrightarrow$ antiparticle. Transformations are now independent of x:

$$\psi \to \psi' = C\,\bar{\psi}^T$$
$$\bar{\psi} \to \bar{\psi}' = -\psi^T C^{-1}.$$

The minus-sign is related to the anticommuting nature of fermions. Then, like with parity:

(a) $$\psi'' = C\,\bar{\psi}'^T = C\,(-\psi^T C^{-1})^T = -C(C^{-1})^T\,\psi \overset{!}{=} \psi$$

$$\Rightarrow\ -C\,(C^T)^{-1} = \mathbb{1} \quad \Rightarrow \quad \boxed{C = -C^T}$$

(b) $$S' = \int_x -\psi^T C^{-1}\,\{i\gamma^\mu\partial_\mu - m\}\,C\,\bar{\psi}^T$$

$v^T A B w = v_\alpha A_{\alpha\beta} B_{\beta\gamma} w_\gamma = v_\alpha A^T_{\beta\alpha} B^T_{\gamma\beta} w_\gamma$
$= w_\gamma B^T_{\gamma\beta} A^T_{\beta\alpha} v_\alpha \times (-1)$
from $v_\alpha w_\gamma = -w_\gamma v_\alpha$

$$= \int_x \bar{\psi}\, C^T\,\{i(\gamma^\mu)^T \overleftarrow{\partial}_\mu - m\}(C^T)^{-1}\,\psi$$

partial integration

$$= \int_x \bar{\psi}\,\{-i(C^T)(\gamma^\mu)^T(C^T)^{-1}\partial_\mu - m\}\,\psi$$

$$\Rightarrow \boxed{C^T(\gamma^\mu)^T(C^T)^{-1} = -\gamma^\mu}$$

We see that a symmetry exists if

$$\begin{cases} C = -C^T \\ C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \end{cases}.$$

Exercise 4.2 $\Rightarrow$ solutions can be found!

Remarks: * Not every action is symmetric under C! In particular:

$$S_5' = \int_x \bar{\psi}'\,\{i\gamma^\mu\partial_\mu - m\}\,\gamma^5\psi'$$
$$= \int_x -\psi^T C^{-1}\,\{i\gamma^\mu\partial_\mu - m\}\gamma^5 C\,\bar{\psi}^T$$
$$= \int_x \bar{\psi}\, C^T (\gamma^5)^T \{\, i(\gamma^\mu)^T\overleftarrow{\partial}_\mu - m\}\,(C^T)^{-1}\psi$$

$\{(\gamma^5)^T, (\gamma^\mu)^T\} = 0$;
$C^T(\gamma^5)^T(C^T)^{-1} = C^T i(\gamma^3)^T(\gamma^2)^T(\gamma^1)^T(\gamma^0)^T(C^T)^{-1}$
$= i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$

partial integration

$$= \int_x \bar{\psi}\,\{i\,C^T(\gamma^\mu)^T(C^T)^{-1}\overrightarrow{\partial}_\mu - m\}\,\gamma^5\psi$$
$$= \int_x \bar{\psi}\,\{-i\gamma^\mu\partial_\mu - m\}\,\gamma^5\psi.$$

* Charge conjugation is different from "Hermitean conjugation", in which "i" changes as well. For instance:

$$S^\dagger = \int_x [\psi^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\psi]^\dagger$$
$$= \int_x \psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m)\,\gamma^{0\dagger}\psi$$

$\gamma^{\mu\dagger}\gamma^{0\dagger} = \gamma^0\gamma^\mu$ & partial integration

$$= \int_x \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi\,;$$

$$S_5^\dagger = \int_x [\psi^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\gamma^5\psi]^\dagger$$
$$= \int_x \psi^\dagger\gamma^{5\dagger}(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m)\,\gamma^{0\dagger}\psi$$

$(\gamma^5)^\dagger = \gamma^5$ & partial integration

$$= \int_x \bar{\psi}\,(i\gamma^\mu\partial_\mu + m)\,\gamma^5\psi.$$

(ii) Conserved particle number

If the action is invariant in a continuous phase transformation,

$$\psi \to \psi' = e^{i\alpha}\psi$$
$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\, e^{-i\alpha},$$

then the Noether theorem guarantees the existence of a conserved particle number. This is the case e.g. with muons in QED: $\mathcal{L} = \ldots + \bar{\psi}_\mu (i\gamma^\mu D_\mu - m_\mu)\psi_\mu$.

Physically, the conservation law exists because at each vertex there are two fermions, so that the number of fermion lines remains unchanged:  .

(iii) Flavour symmetries

Sometimes we can enlarge the symmetry by mixing various fermions; an enlarged symmetry leads to stronger consequences.

For instance: neglecting the mass difference of u- and d-quarks, as well as QED-effects compared with QCD effects, the theory possesses an isospin-symmetry (cf. sec. 4.1):

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} u' \\ d' \end{pmatrix} = \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{R} \begin{pmatrix} u \\ d \end{pmatrix}$$

Invariance of action $\left(S = \ldots - \int_x (\bar{u}\ \bar{d}) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \right)$ requires that $R^\dagger R = \mathbb{1}$, i.e. $R \in U(2)$. One can „factorize" $U(2) \cong U(1) \times SU(2)$; $U(1)$ corresponds to a common phase transformation like above, and $SU(2)$ is an additional non-trivial symmetry.

$\cong$ denotes „isomorphism"

Implications:

* Representations of $SU(2)$ are known from the case of rotations! So we can represent $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}\tfrac{1}{2}\rangle$, $d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$, and their bound states (hadrons) also have specific quantum numbers I, I_z. In particular, all states with the same I have a common mass.

* If a decay goes through strong interactions, I and I_z are conserved. In weak interactions, isospin is not conserved; nevertheless, all final states with the same I lead to related amplitudes.

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