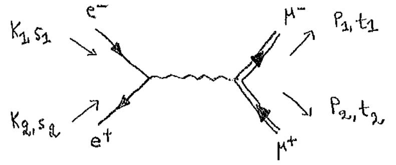
2.2 Muon pair production

We continue with $e^-e^+ \to \mu^-\mu^+$, viz.



p.7 $\Rightarrow$ $\mathcal{M} = \pm\, \bar{v}(\vec{k}_2,s_2)(ie\gamma^\mu)u(\vec{k}_1,s_1)\; \bar{U}(\vec{p}_1,t_1)(ie\gamma^\nu)V(\vec{p}_2,t_2)\; \dfrac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i0^+}$.

The next goal is to compute the corresponding cross section (p.3).

Steps: (i) Since $(p_1+p_2)^2$ is a fixed quantity rather than an integration variable, there is no need for the „$+i0^+$" here.

(ii)
$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}\; \bar{v}(\vec{k}_2,s_2)\gamma^\mu u(\vec{k}_1,s_1)\left[\bar{v}(\vec{k}_2,s_2)\gamma^\alpha u(\vec{k}_1,s_1)\right]^*$$
$$\times\; \bar{U}(\vec{p}_1,t_1)\gamma_\mu V(\vec{p}_2,t_2)\left[\bar{U}(\vec{p}_1,t_1)\gamma_\alpha V(\vec{p}_2,t_2)\right]^*$$

Here:
$$[\bar{v}\gamma^\alpha u]^* = [v^\dagger\gamma^0\gamma^\alpha u]^\dagger = u^\dagger(\gamma^\alpha)^\dagger(\gamma^0)^\dagger v \;;$$
$$(\gamma^0)^\dagger = \gamma^0 \;; \qquad (\gamma^\alpha)^\dagger\gamma^0 = \gamma^0\gamma^\alpha$$
$$\Rightarrow [\bar{v}\gamma^\alpha u]^* = u^\dagger\gamma^0\gamma^\alpha v = \bar{u}\gamma^\alpha v \,.$$

$$= \frac{e^4}{(p_1+p_2)^4}\; \bar{v}(\vec{k}_2,s_2)\gamma^\mu u(\vec{k}_1,s_1)\,\bar{u}(\vec{k}_1,s_1)\gamma^\alpha v(\vec{k}_2,s_2)$$
$$\times\; \bar{U}(\vec{p}_1,t_1)\gamma_\mu V(\vec{p}_2,t_2)\bar{V}(\vec{p}_2,t_2)\gamma_\alpha U(\vec{p}_1,t_1)\,.$$

(iii) If spins are not observed, we can <u>sum over</u> the spins of the final state, and <u>average over</u> the spins of the initial state. This can be achieved with the completeness relations of p. 6:

$$\sum_{s_1} u_\alpha(\vec{k}_1,s_1)\,\bar{u}_\beta(\vec{k}_1,s_1) = (\not{k}_1+m_e)_{\alpha\beta}$$
$$\sum_{s_2} \bar{v}_\gamma(\vec{k}_2,s_2)\,M_{\gamma\delta}\,v_\delta(\vec{k}_2,s_2) = (\not{k}_2-m_e)_{\delta\gamma}M_{\gamma\delta} = \mathrm{Tr}\left[(\not{k}_2-m_e)M\right]$$

The same works with the muon spinors, with $m_e \to m_\mu$.

$$\Rightarrow \underbrace{\frac{1}{4}}_{\text{average over initial spins}}\sum|\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\,\mathrm{Tr}\left[(\not{k}_2-m_e)\gamma^\mu(\not{k}_1+m_e)\gamma^\alpha\right]\mathrm{Tr}\left[(\not{p}_1+m_\mu)\gamma_\mu(\not{p}_2-m_\mu)\gamma_\alpha\right].$$

(iv) The traces over the γ-matrices can be carried out:

(a) $\mu \neq \nu \quad \Rightarrow \quad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \Rightarrow \quad \gamma^\nu = -(\gamma^\mu)^{-1}\gamma^\nu\gamma^\mu$

$$\Rightarrow \quad \mathrm{Tr}(\gamma^\nu) = -\mathrm{Tr}\left[(\gamma^\mu)^{-1}\gamma^\nu\gamma^\mu\right] = -\mathrm{Tr}(\gamma^\nu) \quad \Rightarrow \quad \boxed{\mathrm{Tr}(\gamma^\nu) = 0.}$$

(b) $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\cdot \mathbb{1}_{4\times 4} \quad \Rightarrow \quad \boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}.}$

(c) $\boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0}$, because: * two indices equal, e.g. $\mu=\nu$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = \mathrm{Tr}(\gamma^\rho) = 0$$

* all indices different $\quad \lambda \neq \mu,\nu,\rho$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^\lambda)$$

$$= \mathrm{Tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho).$$

(d) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda) = -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\lambda) + 2\eta^{\mu\nu}\,\mathrm{Tr}(\gamma^\rho\gamma^\lambda)$

$\hookrightarrow \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\lambda) - 2\eta^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\lambda)$

$\hookrightarrow -\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^\mu) + 2\eta^{\mu\lambda}\,\mathrm{Tr}(\gamma^\nu\gamma^\rho)$

$$\Rightarrow \boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda) = 4\left(\eta^{\mu\nu}\eta^{\rho\lambda} - \eta^{\mu\rho}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\rho}\right)}$$

So now:
$$\frac{1}{4}\sum |M|^2 = \frac{4e^4}{(p_1+p_2)^4}\left\{k_2^\mu k_1^\alpha + k_2^\alpha k_1^\mu - \eta^{\mu\alpha}(k_1\cdot k_2 + m_e^2)\right\}$$
$$\times \left\{p_{1\mu}p_{2\alpha} + p_{1\alpha}p_{2\mu} - \eta_{\mu\alpha}(p_1\cdot p_2 + m_\mu^2)\right\}.$$

(v) Space-time indices can be contracted. Using $\eta^{\mu\alpha}\eta_{\mu\alpha} = \eta^\mu{}_\mu = 4$, one gets

$$\frac{1}{4}\sum|M|^2 = \frac{4e^4}{(p_1+p_2)^4}\Big\{2k_2\cdot p_1\, k_1\cdot p_2 + 2k_2\cdot p_2\, k_1\cdot p_1 - 2k_1\cdot k_2\,(p_1\cdot p_2 + m_\mu^2)$$
$$- 2p_1\cdot p_2\,(k_1\cdot k_2 + m_e^2) + 4(k_1\cdot k_2 + m_e^2)(p_1\cdot p_2 + m_\mu^2)\Big\}$$

$$= \frac{4e^4}{(p_1+p_2)^4}\Big\{2k_2\cdot p_1\, k_1\cdot p_2 + 2k_2\cdot p_2\, k_1\cdot p_1$$
$$+ 2k_1\cdot k_2\, m_\mu^2 + 2p_1\cdot p_2\, m_e^2 + 4m_e^2 m_\mu^2\Big\}.$$

(vi) Because $k_1 + k_2 = p_1 + p_2$, not all momenta are independent. Let us re-express the result in terms of „kinematic invariants":

$$s := (p_1+p_2)^2 = (k_1+k_2)^2,$$
$$t := (k_1-p_1)^2 = (k_2-p_2)^2,$$
$$u := (k_1-p_2)^2 = (k_2-p_1)^2.$$

(Actually only two among these are independent:

$$t+u = \frac{1}{2}\left[(k_1-p_1)^2 + (k_1-p_2)^2 + (k_2-p_2)^2 + (k_2-p_1)^2\right]$$
$$= \frac{1}{2}\left[2k_1^2 + p_1^2 + p_2^2 - 2k_1\cdot(p_1+p_2) + 2k_2^2 + p_1^2 + p_2^2 - 2k_2\cdot(p_1+p_2)\right]$$
$$= k_1^2 + k_2^2 + p_1^2 + p_2^2 - s.$$

Then:

$$
\begin{aligned}
2k_2\cdot p_1 &= k_2^2 + p_1^2 - (k_2-p_1)^2 = m_e^2+m_\mu^2 - u = 2k_1\cdot p_2 \\
2k_2\cdot p_2 &= k_2^2 + p_2^2 - (k_2-p_2)^2 = m_e^2+m_\mu^2 - t = 2k_1\cdot p_1 \\
2k_1\cdot k_2 &= (k_1+k_2)^2 - k_1^2 - k_2^2 = s - 2m_e^2 \\
2p_1\cdot p_2 &= (p_1+p_2)^2 - p_1^2 - p_2^2 = s - 2m_\mu^2
\end{aligned}
$$

$$
\Rightarrow \frac{1}{4}\sum|\mathcal{M}|^2 = \frac{2e^4}{s^2}\left\{(m_e^2+m_\mu^2-u)^2 + (m_e^2+m_\mu^2-t)^2 + (s-2m_e^2)2m_\mu^2 + (s-2m_\mu^2)2m_e^2 + 8m_e^2m_\mu^2\right\}
$$

$$
= \frac{2e^4}{s^2}\left\{(m_e^2+m_\mu^2-u)^2 + (m_e^2+m_\mu^2-t)^2 + 2(m_e^2+m_\mu^2)s\right\}.
$$

(vii) This can be inserted into <u>Fermi's Golden Rule</u> from p. 3:

$$
d\sigma(e^-e^+\to\mu^-\mu^+) = \frac{1}{4\sqrt{(k_1\cdot k_2)^2 - k_1^2k_2^2}}\, d\Phi_2 \cdot \frac{1}{4}\sum|\mathcal{M}|^2 .
$$

The flux factor can be expressed as

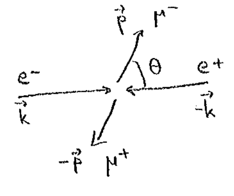
$$
4\sqrt{(k_1\cdot k_2)^2 - k_1^2k_2^2} = 2\sqrt{(s-2m_e^2)^2 - 4m_e^4} = 2\sqrt{s(s-4m_e^2)},
$$

so

$$
d\sigma = \frac{e^4}{s^2\sqrt{s(s-4m_e^2)}} \times \frac{d^3\vec{p}_1}{(2\pi)^3 2E_{p_1}} \times \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}} \times (2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)
$$
$$
\times \left\{(m_e^2+m_\mu^2-u)^2 + (m_e^2+m_\mu^2-t)^2 + 2(m_e^2+m_\mu^2)s\right\}.
$$

(viii) Suppose that we choose the <u>center-of-mass</u> frame:

$$
\vec{k}_1+\vec{k}_2 = \vec{0}
$$
$$
\Rightarrow \vec{p}_1+\vec{p}_2 = \vec{0}
$$



In this frame, let us denote $\vec{k} := \vec{k}_1$, $\vec{p} := \vec{p}_1$, $\cos\theta := \frac{\vec{k}\cdot\vec{p}}{kp}$, as well as $E_k := \sqrt{k^2+m_e^2}$, $E_p := \sqrt{p^2+m_\mu^2}$. Then:

$$
\begin{aligned}
s &= (k_1+k_2)^2 = (2E_k)^2 = 4E_k^2, \\
t &= (k_1-p_1)^2 = m_e^2+m_\mu^2 - 2(E_kE_p - \vec{k}\cdot\vec{p}), \\
u &= (k_1-p_2)^2 = m_e^2+m_\mu^2 - 2(E_kE_p + \vec{k}\cdot\vec{p}).
\end{aligned}
$$

Furthermore energy conservation sets $2E_k = 2E_p$, i.e. $E_k = E_p$. Therefore all quantities can be expressed in terms of <u>two</u> variables, e.g. the initial energy E_k and the scattering angle θ.

(ix) We can then integrate over $\vec{p}_2$ and $p := |\vec{p}|$, following p. 4: Keeping E_k and θ fixed, $\sum |M|^2$ is constant, and we get

$$d^3\vec{p}_1 =: d\Omega \; p^2 \, dp$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{s^2\sqrt{s(s-4m_e^2)}} \left\{ (m_e^2+m_\mu^2-u)^2 + (m_e^2+m_\mu^2-t)^2 + 2(m_e^2+m_\mu^2)s \right\}$$

$$\times \underbrace{\int_0^\infty \frac{dp\, p^2}{(2\pi)^3 2E_p} \cdot \int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}} (2\pi)^4 \delta(2E_k - E_p - E_{p_2}) \, \delta^{(3)}(-\vec{p}-\vec{p}_2)}_{\text{p.4 with } m \to m_\mu;\ M \to 2E_k = \sqrt{s};\ \text{divide by } \int d\Omega = 4\pi}$$

$$\Rightarrow \frac{1}{4\pi} \cdot \frac{1}{8\pi} \, \theta(\sqrt{s} - 2m_\mu) \sqrt{1 - \frac{4m_\mu^2}{s}}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha_{em}^2}{2s^3} \times \theta(\sqrt{s}-2m_\mu) \times \sqrt{\frac{s-4m_\mu^2}{s-4m_e^2}} \times \left\{ (m_e^2+m_\mu^2-u)^2 + (m_e^2+m_\mu^2-t)^2 + 2(m_e^2+m_\mu^2)s \right\},$$

where $\alpha_{em} := \dfrac{e^2}{4\pi}$.

Why is this important?

On p. 3 it was stated that in experiment one would measure the ratio (outcoming rate) / (ingoing rate). But how to measure precisely the ingoing rate, when detectors are on the outside? Indeed it is better to compare two different outcoming rates with each other, because then the ingoing rate drops out:

$$\frac{(\text{outcoming}')}{(\text{outcoming})} = \frac{(\text{outcoming}')/(\text{ingoing})}{(\text{outcoming})/(\text{ingoing})}.$$

Particularly important:

$$\boxed{R(s) := \frac{\sigma(e^-e^+ \to \text{anything})}{\sigma(e^-e^+ \to \mu^-\mu^+)}}.$$

This leads for instance to an experimental determination of the number of colours in QCD, see sec. 4.