2. QED

2.1 Feynman rules

The goal of this section is to „recall" (not to „derive") what the levels (ii),(iii) of p.3 (invariant amplitude, Green's functions) amount to in Quantum Electrodynamics (QED). The resulting recipe is then summarized as „Feynman rules".

We consider QED with two charged particles, electrons and muons:

$$\mathcal{L} := -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}_e \left(i\gamma^\mu D_\mu - m_e \right) \Psi_e + \bar{\Psi}_\mu \left(i\gamma^\mu D_\mu - m_\mu \right) \Psi_\mu + \underbrace{\mathcal{L}_{g.f.}}_{\boxed{\text{gauge fixing}}} ;$$

$$D_\mu := \partial_\mu - ieA_\mu \quad ; \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad \{\gamma^\mu, \gamma^\nu\} := 2\eta^{\mu\nu} .$$

Level (iii): Green's functions

Consider the theory in path integral formalism, with weight $\exp(iS) = \exp(i \int_x \mathcal{L})$; $\int_x := \int dt \int d^3x$, and in Fourier representation:

$$A_\mu(x) = \int_P A_\mu(P) e^{-iP\cdot X} \quad ; \quad \int_P := \int \frac{d^4P}{(2\pi)^4}$$

$$\Psi_e(x) = \int_P \Psi_e(P) e^{-iP\cdot X} \quad ; \quad \bar{\Psi}_e(x) = \int_P \bar{\Psi}_e(P) e^{+iP\cdot X}$$

Quadratic part of the action:

$$* \quad \int_X -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \int_X \int_P \int_Q \frac{1}{2} P_\mu \eta_\nu{}^\alpha (Q^\mu \eta^{\nu\beta} - Q^\nu \eta^{\mu\beta}) \; A_\alpha(P) A_\beta(Q) e^{-i(P+Q)\cdot X}$$

$$= \int_P \int_Q \not{\delta}(P+Q) \left(-\frac{1}{2}\right) A_\alpha(P) \left(P^2 \eta^{\alpha\beta} - P^\alpha P^\beta \right) A_\beta(Q)$$

$$\not{\delta}(P) := \int_X e^{iP\cdot X} = (2\pi)^4 \delta^{(4)}(P);$$

$$\int_P \not{\delta}(P) = 1 .$$

$$* \quad \int_X \bar{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi$$

$$= \int_X \int_P \int_Q \bar{\Psi}(P) \left(-i\gamma^\mu iQ_\mu - m\right) \Psi(Q) e^{i(P-Q)\cdot X}$$

$$= \int_P \int_Q \not{\delta}(P-Q) \; \bar{\Psi}(P) (\not{P} - m) \Psi(Q) ; \qquad \not{P} := \gamma^\mu P_\mu$$

Propagators:

$$\frac{\int_{-\infty}^{\infty} dx \, x^2 e^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}ax^2}} = -2 \frac{d}{da} \ln\left(\sqrt{\frac{2\pi}{a}}\right) = \frac{1}{a}$$

$$\Leftrightarrow \quad \underbrace{A_\alpha(P) A_\beta(Q)}_{} = (\text{Exercise } 2.1) = \not{\delta}(P+Q) \left(\frac{-i\eta_{\alpha\beta}}{P^2 + i0^+} + \ldots \right)$$

$$\underbrace{\Psi(P) \bar{\Psi}}_{}(Q) = (\text{Exercise } 2.1) = \not{\delta}(P-Q) \frac{(i)(\not{P}+m)}{P^2 - m^2 + i0^+} ,$$

where „$+i0^+$" is related to boundary conditions satisfied by the propagator (Exercise 2.2).

Interactions:

$$e^{iS_I} = 1 + iS_I - \frac{1}{2} S_I^2 + \ldots ,$$

and make use of Wick's theorem!

Level (ii): Towards invariant amplitude

For this level we need to recall free relativistic quantum mechanics, in particular the solutions of Dirac and Maxwell equations. Subsequently proper („second quantized") interpretation applies in QED.

(a) Dirac equation:

$$(i\gamma^\mu \partial_\mu - m_e)\,\underset{}{\overset{\text{„second quantized" operator}}{\hat{\Psi}_e}} = 0 .$$

We solve this in momentum space.

$\Rightarrow$ non-trivial solutions exist if $\det(\not{p} - m_e) = 0$

$\Rightarrow$ $p^2 - m_e^2 = 0$!

Denote now $P := (E_p, \vec{p}) := (\sqrt{p^2 + m_e^2}, \vec{p})$.

Then start anew with an „on-shell" ansatz:

$$\hat{\Psi}_e := \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=\pm1} \left[\hat{a}_{\vec{p},s}\, u(\vec{p},s)\, e^{-iP\cdot X} + \hat{b}^{+}_{\vec{p},s}\, v(\vec{p},s)\, e^{iP\cdot X} \right],$$

$$\hat{\bar{\Psi}}_e = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=\pm1} \left[\hat{a}^{+}_{\vec{p},s}\, \bar{u}(\vec{p},s)\, e^{iP\cdot X} + \hat{b}_{\vec{p},s}\, \bar{v}(\vec{p},s)\, e^{-iP\cdot X} \right].$$

Here: $(\not{p} - m_e)\,u = (\not{p} + m_e)\,v = 0$ („Dirac equation")

$$\left.\begin{aligned} \sum_{s=\pm1} u_\alpha \bar{u}_\beta &= (\not{p} + m_e)_{\alpha\beta} \\ \sum_{s=\pm1} v_\alpha \bar{v}_\beta &= (\not{p} - m_e)_{\alpha\beta} \end{aligned}\right\} \text{(„completeness")}$$

$$\{\hat{a}_{\vec{p},s}, \hat{a}^{+}_{\vec{p}',s'}\} = \delta^{(3)}(\vec{p} - \vec{p}')\,\delta_{ss'} = \{\hat{b}_{\vec{p},s}, \hat{b}^{+}_{\vec{p}',s'}\}$$

(„second quantization")

(b) Maxwell equations (in vacuum):

$$\partial_\mu F^{\mu\nu} = 0 \quad \Rightarrow \quad (\partial\cdot\partial\, \eta^\nu{}_\mu - \partial^\nu \partial_\mu)\hat{A}^\mu = 0$$

Again we work in momentum space.
Now a non-trivial solution always exists ($A^\mu \propto p^\mu$), but can be removed by gauge fixing, e.g. Lorenz ($\partial_\mu A^\mu = 0$) or Coulomb ($\nabla\cdot\vec{A} = 0 = \underset{\uparrow \atop \boxed{\text{„tricky"}}}{A^0}$)
Then, „on-shell modes" have $P^2 = 0$.

On-shell ansatz:

$$\hat{A}^\nu := \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2p}} \sum_\lambda \left[\hat{a}_{\vec{p},\lambda}\, \varepsilon^\nu(\vec{p},\lambda)\, e^{-iP\cdot X} + \hat{a}^{+}_{\vec{p},\lambda}\, \varepsilon^{\nu *}(\vec{p},\lambda)\, e^{iP\cdot X} \right]$$

Here: $P\cdot\varepsilon(\vec{p},\lambda) = 0$ („transversality")*

$$\sum_\lambda \varepsilon^\mu \varepsilon^{\nu *} \underset{\uparrow \atop \boxed{\text{depends on gauge}}}{=} -\eta^{\mu\nu} + \frac{p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu}{p\cdot\bar{p}} \;;\; \bar{p} := (p, -\vec{p})$$

(„completeness")

$$[\hat{a}_{\vec{p},\lambda}, \hat{a}^{+}_{\vec{p}',\lambda'}] = \delta^{(3)}(\vec{p} - \vec{p}')\,\delta_{\lambda\lambda'}$$

(„second quantization")

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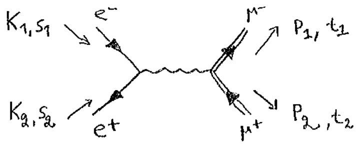
* This comes from the gauge fixing condition.

Putting (ii) and (iii) together



Consider the scattering $e^-(K_1) + e^+(K_2) \to \mu^-(P_1) + \mu^+(P_2)$.

Initial state: $\underset{\text{positron}}{\hat{b}^+_{\vec{k}_2,s_2}}\ \underset{\text{electron}}{\hat{a}^+_{\vec{k}_1,s_1}} |0\rangle$

Final state: $\underset{\text{antimuon}}{\hat{B}^+_{\vec{p}_2,t_2}}\ \underset{\text{muon}}{\hat{A}^+_{\vec{p}_1,t_1}} |0\rangle$

Amplitude: $\langle 0 | \hat{A}_{\vec{p}_1,t_1} \hat{B}_{\vec{p}_2,t_2}\ \underbrace{\hat{U}_I(\infty,-\infty)}\ \hat{b}^+_{\vec{k}_2,s_2} \hat{a}^+_{\vec{k}_1,s_1} |0\rangle$.

time evolution operator:

$$\hat{T}\left\{\exp\left[-i\int_{-\infty}^{\infty} dt\, \hat{H}_I(t)\right]\right\}$$

$$\overset{L_I = -H_I}{=} \underset{\text{time ordering}}{\hat{T}}\left\{\exp\left[i\hat{S}_I\right]\right\}.$$

To get a non-zero amplitude we need one $\hat{a}_{\vec{k}_1,s_1}$ (from $\hat{\Psi}_e$), $\hat{b}_{\vec{k}_2,s_2}$ ($\hat{\bar{\Psi}}_e$), $\hat{A}^+_{\vec{p}_1,t_1}$ ($\hat{\bar{\Psi}}_\mu$), and $\hat{B}^+_{\vec{p}_2,t_2}$ ($\hat{\Psi}_\mu$). These come from second order:

$$\exp(iS_I) = \exp\left\{ i\left[\int_X \bar{\Psi}_e\, i\gamma^\mu(-ieA_\mu)\Psi_e + \int_X \bar{\Psi}_\mu\, i\gamma^\nu(-ieA_\nu)\Psi_\mu\right]\right\}$$

$$= 1 + \ldots \quad + \frac{1}{2}\cdot 2\cdot \int_X \bar{\Psi}_e \gamma^\mu (ieA_\mu)\Psi_e \int_Y \bar{\Psi}_\mu \gamma^\nu (ieA_\nu)\Psi_\mu + \ldots$$

Since A_μ, A_ν do <u>not</u> appear in external states, we can treat them like the Green's function on p. 5, with „$+i0^+$" corresponding to „$\hat{T}$":

$$\underbrace{A_\mu(X) A_\nu(Y)} = \int_{P,Q} \underbrace{A_\mu(P) A_\nu(Q)}\, e^{-iP\cdot X - iQ\cdot Y} = \int_P e^{iP\cdot(Y-X)} \frac{-i\eta_{\mu\nu}}{P^2 + i0^+} \quad \text{(„Feynman gauge")}$$

For the external states, need to go through a tedious analysis, to replace all creation and annihilation operators through anticommutators.* However, modulo overall signs, the result should be clear:

$$\langle 0 | \hat{A}_{\vec{p}_1,t_1} \hat{B}_{\vec{p}_2,t_2}\ \hat{U}_I(\infty,-\infty)\ \hat{b}^+_{\vec{k}_2,s_2} \hat{a}^+_{\vec{k}_1,s_1} |0\rangle$$

$$= \pm \int_{X,Y}\int_P \bar{v}(\vec{k}_2,s_2)(ie\gamma^\mu)u(\vec{k}_1,s_1)\ \bar{U}(\vec{p}_1,t_1)(ie\gamma^\nu)V(\vec{p}_2,t_2)\cdot \frac{-i\eta_{\mu\nu}}{P^2+i0^+}$$

$$\times \frac{e^{-iK_2\cdot X}}{\sqrt{(2\pi)^3 2E_{k_2}}}\cdot \frac{e^{-iK_1\cdot X}}{\sqrt{(2\pi)^3 2E_{k_1}}}\cdot \frac{e^{iP_1\cdot Y}}{\sqrt{(2\pi)^3 2E_{p_1}}}\cdot \frac{e^{iP_2\cdot Y}}{\sqrt{(2\pi)^3 2E_{p_2}}}\cdot e^{iP\cdot(Y-X)}$$

The integrals yield $\int_P \delta(-P-K_1-K_2)\,\delta(P_1+P_2+P) = \delta(P_1+P_2-K_1-K_2)$.
Together with the square roots this leads (when squared) to $d\Phi_2$.
The rest amounts to the invariant amplitude:**

$$\boxed{\mathcal{M} = \pm\ \underset{\text{positron}}{\bar{v}(\vec{k}_2,s_2)}(ie\gamma^\mu)\underset{\text{electron}}{u(\vec{k}_1,s_1)}\ \underset{\text{muon}}{\bar{U}(\vec{p}_1,t_1)}(ie\gamma^\nu)\underset{\text{antimuon}}{V(\vec{p}_2,t_2)}\ \frac{-i\eta_{\mu\nu}}{(P_1+P_2)^2+i0^+}}$$

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*E.g. $\hat{a}_{\vec{k},s}\hat{a}^+_{\vec{k}_1,s_1}|0\rangle$
$= \{\hat{a}_{\vec{k},s}, \hat{a}^+_{\vec{k}_1,s_1}\}|0\rangle$
$- \hat{a}^+_{\vec{k}_1,s_1}\overset{0}{\cancel{\hat{a}_{\vec{k},s}|0\rangle}}$
$= \delta^{(3)}(\vec{k}-\vec{k}_1)\delta_{ss_1}|0\rangle$

** Sometimes an overall $(+i)$ is also inserted.

Feynman rules

Because an explicit derivation, like the one sketched on p.7, is quite tedious, yet the end result for $\mathcal{M}$ is simple, it is sensible to formulate „shortcut" rules which allow us to write down $\mathcal{M}$ directly:

(1) Draw the diagram, labelling the spins and momenta of each line.

(2) For the inner lines, can use propagators („time-ordered", or „Feynman") as described on p.5.

(3) For the vertices, insert $ie\gamma^\mu$.

(4) At each vertex, there is momentum conservation (this was imposed by the integrals $\int_{X,Y}$ on p.7).

(5) For the external lines, insert „polarization vectors":

Incoming particle: $u(\vec{k}_1, s_1)$
Incoming antiparticle: $\bar{v}(\vec{k}_2, s_2)$
Outgoing particle: $\bar{u}(\vec{p}_1, t_1)$
Outgoing antiparticle: $v(\vec{p}_2, t_2)$
Photon: $\epsilon^\mu(\vec{p}, \lambda)$

(6) For identical particles in the final state, may need $\frac{1}{n!}$ (cf. p. 4)

(7) If spins / polarizations are not observed, we can sum over them in $|\mathcal{M}|^2$. This goes through the completeness relations on p.6, cf. sec. 2.2.

(8) There is an extra minus sign for closed fermion loops.

(9) Integrate over closed loop momenta with the measure $\int_P$.

Unsatisfactory points:

* In an inner fermion line, should one have

$$\frac{(i)(\not{p}+m)}{p^2-m^2+i0^+} \quad \text{or} \quad (p \to -p) \quad \frac{(-i)(\not{p}-m)}{p^2-m^2+i0^+} \quad ?$$

$\Rightarrow$ To be sure, could work out from definitions (p.7), but in the end the intuitive simple guess usually works.

* There are many different conventions regarding i's!

For the future:

In these lectures, we will mostly concentrate on Green's functions („level (iii)"), but it is important to keep in mind how these connect to observable quantities (→ „level (ii)" ⟶ „level (i)") ⇒ sec. 2.2.